Understanding Why $\sqrt{2}$ is Irrational

He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side.
Plato

Proving that $\sqrt{2}$ is irrational is an instructor favorite in many math classes: Number Theory, Introduction to Proofs, Discrete Mathematics, and even Math for Poets. There are two reasons for this

1. It is a beautiful proof with a surprising twist.
2. It has an antique origin which had major philosophical implications.

It is nice to add interest to the result by elaborating on the latter. The Pythagoreans believed that the world could be explained by the ratio of integers based on their discoveries on musical scales, heavenly bodies, and other scientific facts. The isosceles right triangles with legs 1 was considered a simple, uncomplicated object.. The discovery that its diagonal $\sqrt{2}$ is irrational was shocking, comparable to the modern discoveries in physics like relativity and quantum mechanics.

Until recently, I would present the standard proof with smug satisfaction that students would appreciate its beauty and power. Answers to an exam question requiring proof that $\sqrt{3}$ is irrational was disturbing.

Clearly, a deeper exploration of this classic irrationality result is required, either as classroom presentation, homework, or class project. I present my personal examination of the result. How deeply you want to explore in class depends on the class preparation and objectives. We outline the framework of our investigation below (as classroom advice) and provide detailed results and proofs in sections 1 and 2.

Our point of view is that to fully understand a proof

1. You must view it as the tip of an iceberg, a preview of further results.
2. You must understand why the proof fails in false extensions of the result.

The second requirement asks for a counter-example. $\sqrt{6}$ is **not** a counter-example , but $\sqrt{4}$ is.

To address the first requirement, you should state that the square root of **any** prime is irrational as well as the product of distinct primes. Also the n-th root of these numbers are also irrational. The exciting part is that the **same proof**, with slight modifications, works in all these cases.

Depending on the class, you can provide a simple, complete test, based on unique factorization, for when $\sqrt{m}$, and indeed when $\sqrt[n]{m}$ **,** is irrational . You may be proud that you have provided a complete treatment of the irrationality for integers -- but there is surprise insight from high school math (treated in section 2) which deepens our understanding further. The surprise has a lot to say about Mathematics.

All letters (m, n, r, s, a, b) denote positive integers. $\frac{r}{s}$ **always** denotes a fraction in reduced form, i.e., r and s have no (non-trivial) common factor. p and q **always** denote **distinct** primes. *Integer divisibility* is denoted $m \mid n$, meaning there exists an integer x such that n=mx. *Non-divisibility* is denoted $m \nmid \mathrm{n}$.

1. THE STANDARD PROOF AND ITS EXTENSIONS.

The standard proof that $\sqrt{2}$ is irrational requires very little, only

Lemma 1.1. *Every rational number (fraction) can be written in reduced lowest form* $\frac{r}{s}$ where r and s *are relatively prime (have no common divisor).*

Lemma 1.2. *If $p \mid ab$ and $p \nmid a$, then $p \mid b$.*

Corollary 1.3. *If $p \mid a^2$, then $p \mid a$.*

Corollary 1.4. *If $2 \mid ab$ and $2 \nmid a$, then $2 \mid b$.*

Andrews [1] (theorem 2-3 and its corollaries, pp 20-21) has a nice treatment of these facts based on the Euclidean algorithm. In an elementary course, it is not advisable to prove the above. Luckily, students accept them as obvious based on the prime factorization theorem, corollary 1.4 being the folk saying, "Odd times odd is odd."

Theorem 1.5. $\sqrt{2}$ *is irrational.*

(Proof by contradiction) . Assume $\sqrt{2} = \frac{r}{s}$ where r and s have no common divisor. We get a contradiction by showing that $2 \mid r$ and $2 \mid s$.

Squaring both sides of our assumption,

$$r^2 = 2s^2. \quad \text{So, } 2 \mid r^2 \text{ (and by corollary 1.3.) } 2 \mid r.$$

Writing $r = 2t$, $4t^2 = 2s^2$ which yields

$$s^2 = 2t^2. \quad \text{So, } 2 \mid s^2 \text{ yielding } 2 \mid s$$

which contradicts our assumption. ■

Theorem 1.7 below lists some extensions of the basic result (in ascending order of sophistication) that your class may want to consider. The amazing fact is that the proof of theorem 1.5, with slight modifications, works for each extension. Some of the extensions do require some new simple facts:

Lemma 1.6.
A. If $p \mid a_1a_2...a_nb$ *and* $p \nmid a_i (i=1,...,n),$ *then* $p \mid b.$
B. If $p \mid a^n$*, then* $p \mid a.$
C. If p *and* q *are distinct primes, then* $p \nmid$ q.

Henceforth we assume $p, p_1, p_2..., p_n$ *are distinct primes.*

Theorem 1.7.
A. $\sqrt{p}$ *is irrational.*
B. $\sqrt{pq}$ *is irrational.*
C. $\sqrt{p_1p_2....p_n}$ *is irrational.*
D. $\sqrt[n]{p^i}$ *is irrational , 0<i<n.*
E. $\sqrt[n]{p^i p_1p_2....p_n}$ *is irrational, 0<i<n.*

It is remarkable that we can prove theorem 1.7.C with no extra effort. Thereafter, theorem 1.7.E is no longer surprising. We provide a sample proof of the extensions in theorem 1.7 .

Example1.8. $\sqrt[5]{p^3q}$ *is irrational.*

Proof by contradiction . Assume $\sqrt[5]{p^3q} = \frac{r}{s}$. We get a contradiction by showing that $p \mid r$ and $p \mid s.$

Raising both sides of our assumption to power 5,

$$r^5 = p^3qs^5. \qquad \text{So, } p \mid r^5 \text{ (and by lemma 1.6.B.) } p \mid r.$$

Writing $r = pt,$ $p^5t^5 = p^3qs^5$ which yields (Note the importance of 3<5)

$$p^2t^5 = qs^5. \qquad \text{So, } p \mid qs^5.$$

Since $p \nmid$ q, $p \mid s^5$ yielding $p \mid s$ which contradicts our assumption. ■

In most of the above courses, it is best to finish here. However, theorem 1.7.E is very close to the best result which is based on Andrews [1] (theorem 2-5, pp 26)

Theorem 1.9 (Integer prime factorization).

For each integer n>1, there exists primes $p_1 < p_2 < ... < p_{D_n}$ *and exponents* $e_1, e_2, ..., e_{D_n}$ *such that*

$$n = p_1^{e_1} p_2^{e_2} ....p_{D_n}^{e_{D_n}} \quad \textit{where } D_n \textit{ is the number of prime divisors of n;}$$

This factorization of n is unique.

Lemma 1.10. $\sqrt[n]{p_1^{nq+r}p_2^{e_2}....p_n^{e_n}}$ *is rational iff* $\sqrt[n]{p_1^{r}p_2^{e_2}....p_n^{e_n}}$ *is rational.*

Theorem 1.11 (Main radical rationality theorem).

$$\sqrt[n]{p_1^{e_1}p_2^{e_2}....p_n^{e_n}} \text{ is rational iff } n \mid e_i, \ (i=1,...,n).$$

Proof **.** By lemma 1.10 we may assume $0 \le e_i < n \ (i=1,...,n)$. If $e_i \ne 0$ for some i, theorem 1.7.E applies and the expression is irrational. ■

Corollary 1.12. $\sqrt{p_1^{e_1}, p_2^{e_2}..., p_n^{e_n}}$ *is rational iff each* e_i *is even, i.e,* $2 \mid e_i, \ (i=1,...,n)$.

Theorem 1.11s seems like the most that can be said about the irrationality of radicals on the elementary level, lemmas 1.2 and 1.6 being a complete tool kit, **but**

2. THE INTEGER ROOT THEOREM.

Lemma 1.2 , the critical tool of section 1 is a special case of theorem 2-3 of Andrews [1]. Our second approach needs the full theorem.

Lemma 2.1. *If* $s \mid ab$ s and b have no (non-trivial) common factor, then $s \mid a$.

As a justification of polynomial factoring algorithms, students are familiar with Barnett [2] (theorem 7, pp 404),

Theorem 2.2 (Rational root theorem). If reduced fraction $\frac{r}{s}$ is a root of

$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x^1 + a_0 = 0$ $\quad a_n \ne 0,\ a_i$ is an integer, *then*

A. $r \mid a_0$.

B. $s \mid a_n$.

Proof **.** We only need (and prove) part B. Note the similarity to the proofs in Section 1.

$$a_n\left(\frac{r}{s}\right)^n + a_{n-1}\left(\frac{r}{s}\right)^{n-1} + ... + a_1\left(\frac{r}{s}\right)^1 + a_0 = 0 \ . \quad \text{Multiplying by } s^n,$$

$$a_n r^n + a_{n-1}r^{n-1}s + ... + a_1 r s^{n-1} + a_0 s^n = 0.$$

Now, $a_n r^n$ is divisible by *s* since all other terms of the equation are divisible by *s*. This fact (that $s \mid a_n r^n$) yields $s \mid a_n$ by lemma 2.1 since r and s have no common factor. ■

When $a_n = 1$, theorem 2.2 says $s\,|\,1$ which implies $s = 1$. Hence

Theorem 2.3 (Integer root theorem).
Any monic, integral polynomial $x^n + a_{n-1}x^{n-1} + ... + a_1x^1 + a_0 = 0$ *only has integers as its rational roots.*

Corollary 2.4.
A. The only rational roots of $x^n = m$ *are integers.*
B. $\sqrt[n]{m}$ *is rational only if* $\sqrt[n]{m}$ *is an integer.*

Does corollary 2.4 give a practical criterion? When students are asked "How do you know that $\sqrt{2}$ is not a fraction?", they answer, "Easy! $1^2 = 1$ *and* $2^2 = 4$." We formalize this argument by

Theorem 2.5 (Student irrationality criterion).
If there exist consecutive integers i *and* $i+1$ *such that*

$$i^n < m < (i+1)^n, \qquad \textit{then}$$

$\sqrt[n]{m}$ *is irrational .*

3. CRITIQUE OF THE APPROACHES.

Let's compare the approaches in sections 1 and 2. Regarding proofs, the essence of both approaches is lemma 2.1. The proof in section 2 is straightforward, while that in section1 is a proof by contradiction. It should be noted that approach 1 uses only a simple, highly intuitive special case of lemma 2.1.

As an irrationality criterion for $\sqrt[n]{m}$, approach 1 must find the prime factorization of *m;* approach 2 is much easier and preferred by students. In fairness, the irrationality criterion based on prime factorization exponents of theorem 1.11 can't be attained by approach 2.

The two approaches are complementary and both are important. The second approach is deeper and more general, holding promise of further results. Nevertheless, the first approach is best suited for introductory classes because of its simplicity and beauty. It also enables activities that deepen the understanding of the result and its proof.

REFERENCES

1. G. Andrews, Number Theory, 1st Dover ed., Dover Publications, New York, 1994.
2. R.A. Barnett, M.R. Ziegler, K.E. College Algebra,8th ed., McGraw-Hill, Boston, 2008