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USING THE MULTIVARIABLE CHAIN RULE

Finding the derivative of an expression with many occurrences of x is a major application of the multivariable chain rule. The usual computation, using extraneous variables, introduces various side calculations and bookkeeping chores. This paper restates the multivariable chain rule to provide a simple straightforward method of computation. Informally, the computation states that

The derivative of an expression with many occurrences of x is the sum of the derivatives of the expression with respect to each separate occurrence of x.

Thomas and Finney [1] (pp. 600–605) contains a standard statement of the multivariable chain rule and its application to computing derivatives.

We consider functions $f(x_1, \ldots, x_n)$ and are interested in $Df(x, \ldots, x)$ where D denotes differentiation with respect to x. Since $\partial x_i / \partial x = 1$ $(i = 1, \ldots, n)$,

$$Df(x, \ldots, x) = \sum \left. \frac{\partial f}{\partial x_i} \right|_{(x, \ldots, x)}.$$

When computing each $\partial f / \partial x_i$ the other variables are kept constant. This is signified by replacing the other variables by c, x being resubstituted for c after the differentiation.

With this agreement about the relation between x and c, we obtain the following application of the multivariable chain rule to functions of the form $f(x, \ldots, x)$.

THEOREM. *If each of*

$$Df(x, c, \ldots, c), Df(c, x, \ldots, c), \ldots, Df(c, c, \ldots, x)$$

exists and is continuous in an interval containing c, then

$$Df(x, x, \ldots, x) = Df(x, c, \ldots, c) + Df(c, x, \ldots, c) + \cdots + Df(c, c, \ldots, x).$$

The usual product and quotient rules for derivatives easily follow from the Theorem. In addition, an "exponential rule" for $Df(x)^{g(x)}$ can be obtained. Here are some applications of the Theorem.

EXAMPLES:

(1)
$$\begin{aligned} Dx e^x / \sin x &= Dx e^c / \sin c + Dc e^x / \sin c + Dc e^c / \sin x \\ &= e^c / \sin c + c e^x / \sin c + c e^c (-1/\sin^2 x) \cos x \\ &= e^x / \sin x + x e^x / \sin x - x e^x \cos x / \sin^2 x. \end{aligned}$$

(2)
$$\begin{aligned} Dx^{x^x} &= Dx^{c^c} + Dc^{x^c} + Dc^{c^x} \\ &= c^c x^{(c^c - 1)} + (\ln c) c^{x^c} c x^{c-1} + (\ln c) c^{c^x} (\ln c\, c^x) \\ &= x^x x^{(x^x - 1)} + x^x x^{x^x} \ln x + x^x x^{x^x} \ln^2 x. \end{aligned}$$

(3)
$$\begin{aligned} D\int_a^{f(x)} g(x, t)\, dt &= D\int_a^{f(x)} g(c, t)\, dt + D\int_a^{f(c)} g(x, t)\, dt \\ &= g(c, f(x)) f'(x) + \int_a^{f(c)} \frac{\partial g}{\partial x}(x, t)\, dt \\ &= g(x, f(x)) f'(x) + \int_a^{f(x)} \frac{\partial g}{\partial x}(x, t)\, dt. \end{aligned}$$

(4)
$$\begin{aligned} D\int_0^x \int_0^x f(u, v)\, du\, dv &= D\int_0^c dv \int_0^x f(u, v)\, du + D\int_0^x dv \int_0^c f(u, v)\, du \\ &= \int_0^c f(x, v)\, dv + \int_0^c f(u, x)\, du \\ &= \int_0^x f(x, v)\, dv + \int_0^x f(u, x)\, du. \end{aligned}$$

The Theorem also applies to expressions involving determinants, inner products, and cross products.