Pease's theorem, a useful formula resurrected

As a highlight of *integration by parts* instructors love to solve $\int e^x \sin(x)dx$. Pease [4] provides a simple formula for this integral as well as integrals like $\int \sin(x)\sin(5x)dx$ and $\int \sin(2x)\cos(3x)dx$.

Pease's formula summarizes the results of some iterated *integration by parts* computations. Despite its simplicity and usefulness, it has not appeared in the standard calculus texts. A possible explanation is that *Tabular Integration* (Thomas [1], pp552) also computes these integrals and is useful elsewhere (reduction formulas).

We hope to popularize Pease's formula by broadening its scope to the class of *quasi – exponential* functions and providing computation aids. We also provide a proof of the formula by an *integration by guessing* technique (Halpern [3]) .

To simplify exposition, all functions mentioned (including derivatives) are continuous. D denotes differentiation with respect to the variable x and we always assume Df exists. Df is also written as f ' or f '(x). $\int$ represents integration and after finding an anti-derivative, we write a final solution with arbitrary constant C. m and n denote positive integers. Thomas [1] is our basic reference for notation, concepts, and techniques. Differentiations like $Dfgh = f'gh + fg'h + fgh'$ are derivable by logarithmic differentiation and even more clearly by Halpern [2].

1. QUASI-EXPONENTIAL FUNCTIONS.

Definition 1. A function $f(x)$ is *quasi-exponential* if $f'' = k_f f$ for some k_f, the *recurrence constant* of $f(x)$.

The basic quasi-exponential functions are $\sin ax$, $\cos ax$, $\sinh ax$, $\cosh ax$, *and* ax. Below are their recurrence constants. Basically, k_f takes on only two values, +1 and -1.

Lemma 1.1.
A. $k_{e^x} = k_{\sinh x} = k_{\cosh x} = 1$ and $k_{\sin x} = k_{\cos x} = -1$. Also $k_x = 0$.
B. $k_{f(ax)} = a^2 k_{f(x)}$ for quasi-exponential $f(x)$.
C. In particular, $k_{e^{ax}} = k_{\sinh ax} = k_{\cosh ax} = a^2$ and $k_{\sin ax} = k_{\cos ax} = -a^2$.
D. $k_f = k_{f'}$.

Using Integration by parts, iterated twice, Pease [4] has shown

Theorem 1.2 (Pease). *If f and g are quasi-exponential, then*

$$\int f \cdot g \quad = \quad \frac{f' \cdot g - f \cdot g'}{k_f - k_g} + C \quad (k_f \neq k_g).$$

Proof (by guessing). There are two natural guesses:

Guess $f' \cdot g$. $\quad Df' \cdot g = f''g + f'g'$.
Guess $f \cdot g'$. $\quad Df \cdot g' = fg'' + f'g'$.

Replacing f'' and g'',

$$Df' \cdot g = k_f\, fg + f'g'.$$
$$Df \cdot g' = k_g\, fg + f'g'.$$

Yielding

$$Df' \cdot g - Df \cdot g' = (k_f - k_g)\mathrm{f} \cdot g.$$
$$f' \cdot g - f \cdot g' \quad = (k_f - k_g)\int \mathrm{f} \cdot g.$$

■

Note that $\int fgh$ is calculated similarly from the guesses: *f ’gh, fg’h, fgh’*, and *f ‘g‘h’.*

Below are some typical uses of theorem 1.2. Similar calculations evaluate $\int \sin(\mathrm{ax})\sin(b\mathrm{x})\mathrm{dx}$, $\int \sin(\mathrm{ax})\cosh(b\mathrm{x})\mathrm{dx}$, and $\int \sinh(\mathrm{ax})\cosh(b\mathrm{x})\mathrm{dx}$.

Example 1.3. $\int e^{ax}\sin(bx)dx \ = \ \dfrac{ae^{ax}\sin(bx) \ - \ be^{ax}\cos(bx)}{a^2 - (-b^2)} + C.$

Example 1.4. $\int \sin(ax)\cdot\cos(bx)dx = \dfrac{a\cdot\cos(ax)\cos(bx) - (-b\cdot\sin(ax)\sin(bx))}{-a^2 - (-b^2)} + C, \ \left(a \neq b\right).$

The below fun corollary follows from $k_x = 0.$

Corollary 1.5 (Pease Lite). *If f is quasi-exponential, then*

$$\int x \cdot f \quad = \quad \frac{xf' - f}{k_f} + C.$$

Example *1.6*. $\int x\sin(ax)\,dx \ = \ \dfrac{ax\cos(ax) - \sin(ax)}{-a^2}$ + C.

REFERENCES

---

1. R.L. Finney, F.R. Giordano, and M.D. Weir, Thomas' Calculus,10th ed., Addison Wesley Longman, Boston, 2001

2. F. Halpern, Using the multivariate chain rule, *Amer. Math. Monthly* **92** (1985) 144-145.

3. F. Halpern, " Integration by guessing." RoyalPathtomath.org, 30 Mar 2017 <http://royalpathtomath.org/docs/Integration%20by%20Guessing.pdf>.

4. D.K. Pease, A useful integral formula, *Amer. Math. Monthly* **66** (1959) 908.