Non-Standard Analysis

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Preliminaries for Calc students

1. The Real Numbers R are a subset of a larger collection of numbers called the Hyper-Reals *R.

2. 'Everything' true about the Reals is also true for the Hyper-Reals:
   Algebra, Number Theory, Trig, Logs and Exponentials, etc.

3. The Hyper-Reals are non-archimedean
   - There are infinitely small numbers called Infinitesimals
   - There are infinitely large integers called Hyper-Integers

Basic Definitions

1. h is <u>Infinitesimal</u> if $\forall n \quad h < 1/n$.

2. x is <u>infinite</u> if $\forall n \; x > n$.

3. x is <u>bounded</u> if $\exists \; r \; x < r$.

4. n is an <u>infinite integer</u> if it is an integer and is infinite.

5. $x \equiv y$ if $|x - y| = h$ for some Infinitesimal h. (x <u>EQUIVALENT</u> to y)

6. The concepts of Hyper-Prime, Hyper-Rational, Hyper-Irrational

7. An <u>algebraic property</u> P(x) is a list of equations and inequalities in x.

Basic Lemmas (x is bounded, *h and h' are* Infinitesimal)

1. n h, $x\,h$, $h + h'$ are Infinitesimal .

2. r is the only real number in the Infinitesimal interval $|x - \mathrm{r}| < h$.

3. If x is bounded, there exists a unique x $\mathrm{x} \equiv x$ (x - x is Infinitesimal)

We write x = st(x), x is the standard part of x.

Note st(x) is the lub of $\{ \mathrm{r} \mid \mathrm{r} < x \}$.

4. (Reflection) If there exists a Hyper-Real in $|x - \mathrm{a}| < \mathrm{c}$ satisfying an algebraic property , then there exists a Real in the interval satisfying it.

5. (Shrinking) If there exists a Real in each interval $|x - \mathrm{a}| < 1/\mathrm{n}$ satisfying an algebraic property , then there exists an Infinitesimal h such that a + h satisfies it.

Continuity, Limit, and Derivative Definitions

1. f is <u>continuous</u> at a if $\forall h \quad f(a) \equiv f(a+h)$, i.e.,

   $a \equiv x$ implies $f(a) \equiv f(x)$.

2. r is the <u>derivative</u> of f at a if $h \quad \dfrac{f(a+h) - f(a)}{h} \qquad r$.

3. $\lim\limits_{x \to a} f(x) = L$ if $\forall h \quad f(a+h) \equiv L$.

Basic Proofs by Calculation

1. f + g is <u>continuous</u> at a if both f and g are.

Proof. $(f+g)(a+h) = f(a+h) + g(a+h) \equiv f(a) + g(a)$

2. $f(x) = x^2$ is <u>continuous</u> at each a

Proof. $f(a+h) = (a+h)^2 = a^2 + 2ah + h^2 \equiv a^2 = f(a)$.

3. The derivative of $f(x) = x^2$ at a is 2a

Proof. $\dfrac{f(a+h) - f(a)}{h} = \dfrac{2ah + h^2}{h} \equiv 2a$

Theorem: (Intermediate Value Theorem)

If $a < b$ and $f(a) < c < f(b)$, $\exists x$ $a < x < b$ with $f(x) = c$.

Proof: Consider $r = \text{lub} \{ x \mid f(x) \leq c \}$.

For all positive h, $f(r + h) > c$. Now by definition of lub, each interval $|x - a| < 1/n$ has an x with $f(x) \leq c$. By Shrinking, $\exists h'$ $f(r + h') \leq c$.

By continuity at r, $f(r) \geq f(r + h)$ and $f(r) \leq f(r + h')$. So, $f(r) = c$.

Theorem: (Zero derivative at a local Maximum)

If $f(x)$ has a local Maximum at a and $f'(a)$ exists, then $f'(a) = 0$.

Proof: If $f'(a) > 0$, then $\forall h$ $\dfrac{f(a + h) - f(a)}{h} > 0$.

Hence for positive h, $f(a + h) > f(a)$. Note $a + h \in$ interval $|x - a| < c$.

By Reflection, $\exists x$ in $|x - a| < c$ with $f(x) > f(a)$, a contradiction.

Integrals

Definition: $\int_a^b f(x)\,dx = I$ if $\forall h$ with b-a / h a hyper-integer

1 $\sum_{i=0}^{i=b-a/h} f(a+ih)\,h$ I and

2 $\sum_{i=0}^{i=b-a/h} f(\underline{x_i})\,h$ I and

3 $\sum_{i=0}^{i=b-a/h} f(\overline{x_i})\,h$ I and

where $f(\underline{x_I})$ is the minimum of f(x) over $a + ih < x < a + (i+1)h$.

4 And also for each infinite integer n and **internal** assignment $\chi : n \text{ ---->} {}^*R$
with $a < \chi_i < \chi_{i+1} \leq b$ ($\chi_0 = a$ and $\chi_n = b$)
the above conditions on the upper and lower Rieman sums 2 & 3 hold.

Theorem:

1. If f(x) is continuous on [a,b] then I exists.

3. If f(x) is monotonic on [a,b] then I exists.

3. For χ_{RAT}, I never exists.

Informal Criteria: $\int f(x)\,dx = I$ exists if $\forall h$, <u>Total Deviation</u> =

$$\sum_{i=\cancel{b-a/h}\;0}^{i=b-a/h} \left(f(\overline{x_i}) - f(\underline{x_i}) \right) \cdot h \quad \text{is always Infinitesimal .}$$

1. For monotonic f(x),

$$\text{Total Deviation} = h \cdot \sum_{i=0}^{i=b-a/h} \left(f(\overline{x_i}) - f(\underline{x_i}) \right) = h\,(b-a)$$

2. If $\forall x$ $\left| f(x+h) - f(x) \right| < B \sqrt[n]{h}$,

$$\text{Total Deviation} < B\,(b-a)\sqrt[n]{h}$$

<u>A Calculation:</u> $\int_0^x t\,dt =$

$$\sum_{i=0}^{i=x/h} i\,h\,h = h^2 \cdot \sum_{i=0}^{i=x/h} i = h^2 \, \frac{x/h\,(x/h+1)}{2} \equiv \frac{x^2}{2}$$

Definition: If f(x) is continuous in [a , b], define a new function on [a , b]

$F(x) = \int_a^x f(t)\,dt$. F(x) is the integral of f(x) based at a.

Theorem: The derivative of the integral of f(x) is f(x). $F'(x) = f(x)$.

Proof: $F'(x) =$

$$\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt = \frac{f(x+h)\,h}{h} \equiv f(x)$$

Fundamental Theorem of Calculus: The integral of a derivative is the original function plus an adjustment constant, i.e

$$F(x) - F(a) = \int_a^x F'(t)\,dt.$$

when F(x) has a derivative everywhere in [a, b] and it is continuous.

(continuity is needed so we know the integral is defined).

Proof:

Define $G(x) = \int_a^x F'(t)\,dt$. $G'(x) = F'(x)$.

So, $G(x) = F(x) + C$ for some constant C.

How do we know how to adjust the constant.

$$F(x) - F(a) =$$

1. $$F(x) - F(x-h) + F(x-h) - F(x-2h) + F(x-2h) - F(x-3h) + \ldots$$

$$F(a+3h) - F(a+2h) + F(a+2h) - F(a+h) + F(a+h) - F(a)$$

2. $$\sum_{i=0}^{i=x-a/h} F(a+(i+1)h) - F(a+ih)$$

3. $$\sum_{i=0}^{i=x-a/h} \frac{F(a+(i+1)h) - F(a+ih)}{h} \; h \quad \overset{?}{\equiv}$$

4. $$\sum_{i=0}^{i=x-a/h} F'(a+ih) \; h =$$

5. $$\int_a^x F'(t)\,dt.$$

Construction of the Hyper-Reals

Essentially, each Hyper-Real is a countable sequence $\langle r_1, r_2, r_3 \ldots\rangle$ of Reals endowed with a tricky rule for when sequences are equal.

Let I be any infinite set (e.g., the Integers, Pairs of Integers, finite sets of Integers, the Reals, the analytic functions). Consider the Boolean Algebra 2^I of all subsets of I .

Definition: $D \subset 2^I$ is an Ultrafilter if

1. $S \in D$ & $S' \in D$ imply $S \cap S' \in D$
2. $S \in D$ & $S \subset S'$ imply $S' \in D$
3. $S \in D$ or complement $(S) \in D$

Let A be a structure (e.g., a group, a ring, a graph (binary relation).

Consider A^I = the set of I sequences of members of A.

Definition: A^I / D is the Ultrapower of A mod D .

$\mathcal{I}$ is the structure defined by

1. The elements are sequences s of A^I

2. $s = s'$ if $\{ i \mid s_i = s'_i \} \in D$

   We say s = s' almost everywhere (a.e.) and call the equivalence class of s the <u>Monad</u> of s.

3. Operations are performed coordinatewise

   e.g, $s + s' = \langle s_i + s'_i \rangle$

4. Relations R are defined by

   $R(s, s')$ iff $\{ i \mid R(s_i, s'_i \} \in D$

Theorem: A first order sentence is true in A iff it is true in A^I / D

We also say there is an elementary embedding $A \longrightarrow A^I / D$.

D - LIMITS

Let *T* be a topological space and *D* an Ultrafilter on I

Definition: For $S \in T^I$, *D*-Lim S = t if for every t-Neighborhood H,

$$\{ i \mid S_i \in H \} \in D .$$

Theorem: 1. If S = S' in T^I / D, then *D*-Lim S = *D*-Lim S' .

2. If T is compact, then , *D*-Lim S always exists.

3. , *D*-Lim S acts just like limits of countable sequences.

Theorem: (Halpern)

1. If *T* is dense in $\overline{T}$, then *D*-Lim maps T^I / D *onto* $\overline{T}$.
2. If *T* is a sub-structure of $\overline{T}$, then *D*-Lim is a structure homomorphism.

Note that *T* and T^I / D satisfy exactly the same sentences.

Application: Rats are dense in the Reals. Consider a sentence

$$\sigma = \forall x \, \exists y \quad P(x, y) = 0 \text{ and } |x| > |y|$$

Rats $\models \sigma$ implies Reals. $\models \sigma$

One suitable definition of "all the sets which arise in classical analysis" begins by defining a superstructure over a set of atoms (or urelements)

$$X_0 \quad \text{an infinite set of atoms}$$

$$X_{k+1} = \mathcal{P}\left(\bigcup_{n=0}^{k} X_n\right), \qquad k \in \mathbb{N},$$

finally,

$$\mathcal{X} = \bigcup_{k \in \mathbb{N}} X_k.$$

We shall begin by looking at the case where $X_0 = \mathbb{R}$, the real numbers (as a set of atoms). Later we need to construct a superstructure $\mathcal{Y}$ on another set of atoms. Real valued functions are *elements* of $\mathcal{X}$, since functions "are" sets of ordered pairs and ordered pairs "are" sets $\{\{a\},\{a,b\}\}$. The euclidean spaces $\mathbb{R}^n$ and all their subsets are also in $\mathcal{X}$ as elements. The classical spaces $C^2(\mathbb{N})$, $L^p[0,1]$, $H^x(U)$, and so on, all are elements of $\mathcal{X}$.

Elements of $\mathcal{X}$ are called *entities* and the set theory of entities provides a framework for classical analysis. The algebraic object $(\mathcal{X}, \in)$ is called a *superstructure*. The bounded first order language of $\in$ describes the set theory of entities where we always have

$$\forall x\,[x \in A \Rightarrow \cdots] \quad \text{or} \quad \exists y\,[y \in B\ \&\ \cdots]$$

with A and B entities of $\mathcal{X}$. We have one constant in our language $L(\in)$ for each element of $\mathcal{X}$. These constants in turn have interpretations back in $\mathcal{X}$, for example, ρ of L might stand for $\mathbb{R} \in \mathcal{X}$ and we name the standard interpretation map "I". $I(\rho) = \mathbb{R}$ in this case. We may think of any property of analysis expressed in terms of set theory and thus in terms of the relations $\in$ and $=$ of the superstructure. This is what we apply nonstandard extension to.

1.1. Leibniz' Principle. *There is a set of atoms Y_0 and a mapping $*: \mathcal{X} \to \mathcal{Y}$, $\mathcal{Y}$ being the superstructure based on Y_0, with the following properties:*

(i) $^*\mathbb{R} = Y_0$, *the map applied to the ground set entity in $\mathcal{X}$ is the ground set in $\mathcal{Y}$.*

(ii) *The extension is proper, the hyperreal numbers $^*\mathbb{R}$ properly contain the standard embedded reals $^\sigma\mathbb{R} = \{^*r : r \in \mathbb{R}\}$, in particular, there is a nonzero infinitesimal in $^*\mathbb{R}$.*

(iii) *Every sentence about $\mathcal{X}$ with a bounded formalization in the language of $\in$ holds in $\mathcal{X}$ if and only if the $*$-transform obtained by extending each constant of the sentence holds in the range $^*\mathcal{X}$.*

$\mathscr{D}$-lim operator

We prove Theorems 1-5 by showing that each situation yields a diagram to which Theorem 6 applies. Let I be a set. $S_\omega(I)$ denotes the set of all *finite* subsets of I. $\mathscr{D}$ is a *regular ultrafilter* on $S_\omega(I)$ if $\mathscr{D}$ is an ultrafilter on $S_\omega(I)$ such that

$$\forall i_0 \in I, \{S \in S_\omega(I) \mid i_0 \in S\} \in \mathscr{D} .$$

It follows that $\{S \in S_\omega(I) \mid T \subset S\} \in \mathscr{D}$ for each finite $T \subset I$.

Assume a topological space X and ultrafilter $\mathscr{D}$ on I. For each $x \in X$ and $f \in X^I$, $x = \mathscr{D}$-limit f if $\{i \mid f(i) \in V_x\} \in \mathscr{D}$ for each open set V_x containing x.

Essentially the $\mathscr{D}$-lim operator on sequences $f \in X^I$ acts like the limit operation on countable sequences, with the added property that, if X is a compact space, $\mathscr{D}$-lim f always exists. The properties of the $\mathscr{D}$-limit operator are listed below. A more complete teatment is provided in Chang and Keisler [3], pgs. 6-15.

Let X be a topological space with basis ∂ (i.e., for each open set V, $V = \bigcup_{0 \in \partial, 0 \subseteq V} 0$) and let $\mathscr{D}$ be an ultrafilter on I. f will vary over elements of X^I.

$\mathscr{D}$-L0 $\mathscr{D}$-lim f is unique.

$\mathscr{D}$-L1 If $\{i \mid f(i) = f'(i)\} \in \mathscr{D}$, then $\mathscr{D}$-lim $f = \mathscr{D}$-lim f' (or both don't exist).

$\mathscr{D}$-L2 $\mathscr{D}$-lim $\langle f_1, \cdots, f_n \rangle = \langle \mathscr{D}\text{-lim } f_1, \cdots, \mathscr{D}\text{-lim } f_n \rangle$.

$\mathscr{D}$-L3 $\mathscr{D}$-lim f belongs to the closure (in X) of the range of f, (range $f = \{x \in X \mid \exists i \in I, f(i) = x\}$).

$\mathscr{D}$-L3' The $\mathscr{D}$-limit of a constant sequence equals that constant value.

$\mathscr{D}$-L4 If F is a continuous function from X to Y, then $\mathscr{D}$-lim $F(f) = F(\mathscr{D}\text{-lim } f)$.

$\mathscr{D}$-L4' If R is a closed n-ary relation and $\forall i \in I$, $R(f_1(i), \cdots, f_n(i))$, then $R(\mathscr{D}\text{-lim } f_1, \cdots, \mathscr{D}\text{-lim } f_n)$.

$\mathscr{D}$-L5 If X is a compact space, then $\forall f \in X^I$ $\mathscr{D}$-lim f exists.

$\mathscr{D}$-L6 If $\mathscr{D}$ is a regular ultrafilter on $I = S_\omega(\partial)$, ∂ a basis for $\bar{X}$, then, for each set $S \subseteq \bar{X}$ and limit point x_0 of S, there exists a sequence $f \in S^I$ such that $\mathscr{D}$-lim $f = x_0$.

$\mathscr{D}$-L6' If X is dense in $\bar{X}$, for each $x_0 \in \bar{X}$, there exists an X-sequence $f \in X^I$ such that $\mathscr{D}$-lim $f = x_0$ (i.e., the map $X^I \xrightarrow{\mathscr{D}\text{-lim}} \bar{X}$ is onto).

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TRANSFER THEOREMS FOR TOPOLOGICAL STRUCTURES

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Transfer theorems are obtained for the following mathematical situations.

$\mathscr{X}$ is a dense substructure of the compact structure $\bar{X}$. $\{\mathscr{X}_i\}$ is the set of all finitely generated substructures of X. F is a structure of functions from Y to the structure X.

The sentences transferred in the above situations are best described as "almost" positive, variables appearing in a negative subformula are quantified in a prescribed manner.

The main tools of this investigation are the manipulation of classical transfer theorems in the context of commutative diagrams, the ultraproduct construction, and the $\mathscr{D}$-limit operation of Chang and Keisler's "Continuous Model Theory."

Introduction. Every subgroup of an abelian group is abelian, any extension ring of a ring with zero divisors has zero divisors, and each homomorphic image of a commutative ring is commutative are instances of classical transfer theorems (Lemma 1). In this paper transfer theorems are obtained for the following mathematical situations:

X is a dense substructure of the compact topological structure $\bar{X}$,

$\{X_i\}$ is the set of all finitely generated substructures of X, and F is a structure of functions from Y to the structure X.

The sentences transferred in the above situations are best described as "almost" positive, variables appearing in a negative subformula of the sentence are quantified in a prescribed manner.

The main tools of this investigation are the manipulation of the classical transfer theorems in the context of commutative diagrams, the ultraproduct construction, and the $\mathscr{D}$-limit operator popularized by Chang and Keisler [3].

Consider a first order language $\mathscr{L}$ and its associated structures. A formula σ is identifined with its prenex normal form, i.e., σ is assumed to be of the form

$$Q_1v_1Q_2v_2, \cdots, Q_nv_nM\,,$$

where each Q_i is a quantifier ($\forall$ or $\exists$), each v_i is a variable, and M is a formula constructed from (positive) atomic formulae and their negations, *negative atomic formulae*, using the connectives "and"

$\mathscr{M}$.

The formula σ is *∃-generalized positive* if the prefix of σ begins with an initial string of existential quantifiers (i.e., $\sigma = \exists\bar{v}Q_i v_i, \cdots, Q_n v_n M$) such that every negative variable of σ is free or existentially quantified in the initial string $\exists\bar{v}$. The negation of a ∃-generalized positive formula is called a *∀-generalized negative formula*: it begins with an initial string of universal quantifiers such that each of it nonnegative variables is free or universally quantified in the initial string.

THEOREM 1. *Let $\mathscr{A}$ be a dense substructure of the compact structure $\bar{\mathscr{A}}$. Every ∃-generalized positive sentence satisfied by $\mathscr{A}$ is satisfied by $\bar{\mathscr{A}}$.*

This generalizes a result of Robinson [7].

THEOREM 2. *Let $\bar{\mathscr{M}}$ compactify the collection of structures $\mathscr{M}$.*

(i) *Every ∃-generalized positive sentence satisfied in $\mathscr{M}$ is satisfied in $\bar{\mathscr{M}}$.*

(ii) *Every ∀-generalized negative sentence valid in $\bar{\mathscr{M}}$ is valid in $\mathscr{M}$.*

COROLLARIES TO THEOREM 2. *Let σ be a ∀-generalized negative sentence* (in an appropriate language).

2.1. *σ is valid for all extensions of the group $\mathscr{G}$ if σ is valid for all compact groups which are extensions of $\mathscr{G}$.*

2.2. *σ is valid for all total orderings if σ is valid for all complete total orderings.*

2.3. *σ is valid for all totally ordered lattices extending the ordering $\mathscr{F}$ if σ is valid for all complete totally ordered lattices extending $\mathscr{F}$.*

2.4. *σ is valid for all distributive lattices if σ is valid for all complete distributive lattices.*

2.5. *σ is valid for all boolean algebras if σ is valid for all complete boolean algebras.*

(ii) *If, in addition, $\mathscr{A}$ is a compact structure, then every ∀⁻∃-generalized positive sentence valid in $\{\mathscr{A}_i\}$ is satisfied by $\mathscr{A}$.*

COROLLARIES TO THEOREM 4. *Let σ be a ∀⁻∃-generalized positive sentence.*

4.1. *If σ is valid for all finitely generated groups, then σ is valid for all compact groups.*

4.2. *If σ is valid for all finite total orderings* (viewed as a lattice), *then σ is valid for all complete totally ordered lattices.*

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More Worthwhile References

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