Non-Standard Analysis

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Preliminaries for Calc students

1. The Real Numbers \( R \) are a subset of a larger collection of numbers called the Hyper-Reals \(*R\).

2. ‘Everything’ true about the Reals is also true for the Hyper-Reals:
   Algebra, Number Theory, Trig, Logs and Exponentials, etc.

3. The Hyper-Reals are non-archimedean
   - There are infinitely small numbers called Infinitesimals
   - There are infinitely large integers called Hyper-Integers

Basic Definitions

1. \( h \) is **Infinitesimal** if \( \forall n \quad h < 1/n. \)

2. \( x \) is **infinite** if \( \forall n \quad x > n. \)

3. \( x \) is **bounded** if \( \exists r \quad x < r. \)

4. \( n \) is an **infinite integer** if it is an integer and is infinite.

5. \( x \equiv y \) if \( |x - y| = h \) for some Infinitesimal \( h. \) (\( x \) **Equivalent to** \( y \))

6. The concepts of Hyper-Prime, Hyper-Rational, Hyper-Irrational

7. An **algebraic property** \( P(x) \) is a list of equations and inequalities in \( x. \)
Basic Lemmas \( (x \text{ is bounded, } h \text{ and } h' \text{ are Infinitesimal}) \)

1. \( n \ h, \ x \ h, \ h + h' \text{ are Infinitesimal}. \)

2. \( r \) is the only real number in the Infinitesimal interval \( |x - r| < h. \)

3. If \( x \) is bounded, there exists a unique \( x \equiv x \) \((x - x \text{ is Infinitesimal})\)

We write \( x = st(x), \ x \text{ is the standard part of } x. \)

Note \( st(x) \) is the lub of \( \{ r | r < x \} \).

4. (Reflection) If there exists a Hyper-Real in \( |x - a| < c \) satisfying an algebraic property, then there exists a Real in the interval satisfying it.

5. (Shrinking) If there exists a Real in each interval \( |x - a| < 1/n \) satisfying an algebraic property, then there exists an Infinitesimal \( h \) such that \( a + h \) satisfies \( st. \)
Continuity, Limit, and Derivative Definitions

1. \( f \) is \textbf{continuous} at \( a \) if \( \forall h \quad f(a) \equiv f(a + h), \) i.e.,
   \[ a \equiv x \quad \text{implies} \quad f(a) \equiv f(x). \]

2. \( r \) is the \textbf{derivative of} \( f \) at \( a \) if \( h \quad f(a + h) - f(a) \quad \frac{r}{h}. \)

3. \( \lim_{x \rightarrow a} f(x) = L \) if \( \forall h \quad f(a + h) \equiv L. \)

Basic Proofs by Calculation

1. \( f + g \) is \textbf{continuous} at \( a \) if both \( f \) and \( g \) are.

   Proof. \( (f + g)(a + h) = f(a + h) + g(a + h) \equiv f(a) + g(a) \)

2. \( f(x) = x^2 \) is \textbf{continuous} at each \( a \)

   Proof. \( f(a + h) = (a + h)^2 = a^2 + 2ah + h^2 \equiv a^2 = f(a). \)

3. The \textit{derivative} of \( f(x) = x^2 \) at \( a \) is \( 2a \)

   Proof. \( \frac{f(a + h) - f(a)}{h} = \frac{2ah + h^2}{h} \equiv 2a \)
Theorem: (Intermediate Value Theorem)

If \( a < b \) and \( f(a) < c < f(b) \), \( \exists x \ a < x < b \) with \( f(x) = c \).

Proof: Consider \( r = \text{lub} \\{ x \mid f(x) \leq c \} \).

For all positive \( h \), \( f(r + h) > c \). Now by definition of lub each interval \( \{ x - a \mid < 1/n \} \) has an \( x \) with \( f(x) \leq c \). By Shrinking, \( \exists h' \ f(r + h') \leq c \).

By continuity at \( r \), \( f(r) \leq f(r + h) \) and \( f(r) \geq f(r + h') \). So, \( f(r) = c \).

Theorem: (Zero derivative at a local Maximum)

If \( f(x) \) has a local Maximum at \( a \) and \( f'(a) \) exists, then \( f'(a) = 0 \).

Proof: If \( f'(a) > 0 \), then \( \forall h \frac{f(a + h) - f(a)}{h} > 0 \).

Hence for positive \( h \), \( f(a + h) > f(a) \). Note \( a + h \) interval \( \{ x - a \mid < c \). By Reflection, \( \exists x \ \{ x - a \mid < c \) with \( f(x) > f(a) \), a contradiction.
Integrals

Definition: \[ \int_a^b f(x) \, dx = \lim_{h \to 0} \sum_{i=0}^{n} f(a + ih) \cdot h \]

1. \[ \sum_{i=0}^{n} f(a + ih) \cdot h \]

2. \[ \sum_{i=0}^{n} f(x_i) \cdot h \]

3. \[ \sum_{i=0}^{n} f(x_i) \cdot h \]

where \( f(x_i) \) is the minimum of \( f(x) \) over \( a + ih < x < a + (i+1)h \).

And also for each infinite integer \( n \) and \textit{internal} assignment \( \chi : n \to \mathbb{R} \) with \( a < \chi_k < \chi_{k+1} < b \) \((\chi_0 = a \text{ and } \chi_n = b)\)

the above conditions on the upper and lower Riemann sums hold.

Theorem:

1. If \( f(x) \) is continuous on \([a,b]\) then \( I \) exists.

2. If \( f(x) \) is monotonic on \([a,b]\) then \( I \) exists.

3. For \( \chi_{\text{RAT}} \), \( I \) never exists.
Informal Criteria: \( f(x) \, dx = 1 \) exists if \( \forall h \), Total Deviation =

\[
i = \frac{b-a}{h} \\
\sum_{i=0}^{n} (f(x_i) - f(x_i)) \cdot h \quad \text{is always Infinitesimal}.
\]

1. For monotonic \( f(x) \),

\[
\text{Total Deviation} = h \cdot \sum_{i=0}^{n} (f(x_i) - f(x_i)) = h(b - a)
\]

2. If \( \forall x \quad |f(x+h) - f(x)| < B \sqrt{h} \),

\[
\text{Total Deviation} < B(b - a) \sqrt{h}
\]

A Calculation:

\[
\int_{0}^{x} t^2 dt = \\
\sum_{i=0}^{n} i^2 h = h^2 \cdot \sum_{i=0}^{n} i = h^2 \frac{x}{h} \left( \frac{x}{h} + 1 \right) = \frac{x^2}{2}
\]
Definition: If \( f(x) \) is continuous in \([a, b]\), define a new function on \([a, b]\)

\[
F(x) = \int_a^x f(t) \, dt. \quad \text{\( F(x) \) is the integral of \( f(x) \) based at \( a \).}
\]

Theorem: The derivative of the integral of \( f(x) \) is \( f(x) \). \( F'(x) = f(x) \).

Proof: \( F'(x) = \)

\[
\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt = \frac{f(x+h) - f(x)}{h} = f(x)
\]

Fundamental Theorem of Calculus: The integral of a derivative is the original function plus an adjustment constant, i.e

\[
F(x) - F(a) = \int_a^x F'(t) \, dt.
\]

when \( F(x) \) has a derivative everywhere in \([a, b]\) and it is continuous.

(continuity is needed so we know the integral is defined).

Proof:

Define \( G(x) = \int_a^x F'(t) \, dt \). \( G'(x) = F'(x) \).

So, \( G(x) = F(x) + C \) for some constant \( C \).

How do we know how to adjust the constant.
\[
F(x) - F(a) =
\]
\[
\sum_{i = 0} \frac{x-a}{h} \frac{F(a + (i+1) \cdot h) - F(a+i \cdot h)}{h}
\]
\[
i = \frac{x-a}{h}
\]
\[
\sum_{i = 0} \frac{F'(a + i \cdot h)}{h} =
\]
\[
\int_{a}^{x} F'(t) \, dt.
\]
Construction of the Hyper-Reals

Essentially, each Hyper-Real is a countable sequence \(<r_1, r_2, r_3, \ldots>\) of Reals endowed with a tricky rule for when sequences are equal.

Let \(I\) be any infinite set (e.g., the Integers, Pairs of Integers, finite sets of Integers, the Reals, the analytic functions). Consider the Boolean Algebra \(2^I\) of all subsets of \(I\).

I

Definition: \(D \subset 2^I\) is an Ultrafilter if

1. \(S \in D\) & \(S' \in D\) imply \(S \cap S' \in D\)
2. \(S \in D\) & \(S \subset S'\) imply \(S' \in D\)
3. \(S \in D\) or complement \((S) \notin D\)

Let \(A\) be a structure (e.g., a group, a ring, a graph (binary relation)).

Consider \(A = \{i\}\) sequences of members of \(A\).

Definition: \(A / D\) is the Ultrapower of \(A\) mod \(D\).

It is the structure defined by
1. The elements are sequences $s$ of $A$

2. $s = s'$ if $\{ i \mid s_i = s'_i \} \subseteq D$

   We say $s = s'$ almost everywhere (a.e.) and call the equivalence class of $s$ the 
   Monad of $s$.

3. Operations are performed coordinatewise

   e.g., $s + s' = <s_1 + s'_1, \ldots, s_n + s'_n>$

4. Relations $R$ are defined by

   $R(s, s')$ iff $\{ i \mid R(s_i, s'_i) \} \subseteq D$

Theorem: A first order sentence is true in $A$ iff it is true in $A \subseteq D$

We also say there is an elementary embedding $A \rightarrow A \subseteq D$. 
$D$ - LIMITS

Let $T$ be a topological space and $D$ an Ultrafilter on $I$

Definition: For $S \in T$, $D$-Lim $S = t$ if for every $t$-Neighborhood $H,$

$$\{ i | S_i \in H \} \in D.$$

Theorem: 1. If $S = S'$ in $T \ / D$, then $D$-Lim $S = D$-Lim $S'$.

2. If $T$ is compact, then $D$-Lim $S$ always exists.


Theorem: (Halpern)

1. If $T$ is dense in $\bar{T}$, then $D$-Lim maps $T \ / D$ onto $\bar{T}$.

2. If $T$ is a sub-structure of $\bar{T}$, then $D$-Lim is a structure homomorphism.

Note that $T$ and $T \ / D$ satisfy exactly the same sentences.

Application: Rats are dense in the Reals. Consider a sentence

$$\sigma = \forall x \exists y \ P(x, y) = 0 \text{ and } |x| > |y|$$

Rats $\models \sigma$ implies Reals $\not\models \sigma$.
One suitable definition of "all the sets which arise in classical analysis" begins by defining a superstructure over a set of atoms (or urelements)
\[ X_n, \quad \text{an infinite set of atoms} \]
\[ X_n = \mathcal{P} \left( \bigcup_{n=1}^{\infty} X_n \right), \quad k \in \mathbb{N}. \]
finally,
\[ \mathcal{Y} = \bigcup_{k=1}^{\infty} X_k. \]

We shall begin by looking at the case where \( X_n = \mathbb{R} \), the real numbers (as a set of atoms). Later we need to construct a superstructure \( \mathcal{Y} \) on another set of atoms. Real valued functions are elements of \( \mathcal{Y} \) since functions "are" sets of ordered pairs and ordered pairs "are" sets \( \{(a), \{a, b\}\} \). The euclidean spaces \( \mathbb{R}^n \) and all their subsets are also in \( \mathcal{Y} \) as elements. The classical spaces \( \mathcal{C}(\mathbb{U}), \mathcal{L}([0, 1]), \mathcal{H}(\mathcal{U}) \), and so on, are all elements of \( \mathcal{Y} \).

Elements of \( \mathcal{Y} \) are called entities and the set theory of entities provides a framework for classical analysis. The algebraic object \( (\mathcal{Y}, \in) \) is called a superstructure. The bounded first order language of \( \in \) describes the set theory of entities where we always have
\[ \forall x [x \in A \Rightarrow \cdots] \quad \text{or} \quad \exists y [y \in B \& \cdots] \]
with \( A \) and \( B \) entities of \( \mathcal{Y} \). We have one constant in our language \( L(\in) \) for each element of \( \mathcal{Y} \). These constants in turn have interpretations back in \( \mathcal{Y} \), for example, \( \rho \) of \( L \) might stand for \( \mathbb{R} \in \mathcal{Y} \) and we name the standard interpretation map "1". \( I(\rho) = \mathbb{R} \) in this case. We may think of any property of analysis expressed in terms of set theory and thus in terms of the relations \( \in \) and \( = \) of the superstructure. This is what we apply nonstandard extension to.

1.1. **Leibniz Principle.** There is a set of atoms \( \mathcal{Y} \), and a mapping \( \ast : \mathcal{Y} \rightarrow \mathcal{Y} \), \( \mathcal{Y} \) being the superstructure based on \( \mathcal{Y} \), with the following properties:

(i) \( \ast \mathbb{R} = \mathbb{R} \), the map applied to the ground set entity in \( \mathcal{Y} \) is the ground set in \( \mathcal{Y} \).

(ii) The extension is proper, the hyperreal numbers \( \ast \mathbb{R} \) properly contain the standard embedded reals \( \mathbb{R} = \{ \ast r : r \in \mathbb{R} \} \), in particular, there is a nonzero infinitesimal in \( \ast \mathbb{R} \).

(iii) Every sentence about \( \mathcal{Y} \) with a bounded formalization in the language of \( \in \) holds in \( \mathcal{Y} \) if and only if the \( \ast \)-transform obtained by extending each constant of the sentence holds in the range \( \ast \mathcal{Y} \).
We prove Theorems 1-5 by showing that each situation yields a diagram to which Theorem 6 applies. Let $I$ be a set. $S_n(I)$ denotes the set of all finite subsets of $I$. $\mathcal{U}$ is a regular ultrafilter on $S_n(I)$ if $\mathcal{U}$ is an ultrafilter on $S_n(I)$ such that

$$\forall i \in I, \exists S \in S_n(I) \mid i \in S \in \mathcal{U}.$$  

It follows that $\{S \in S_n(I) \mid T \subseteq S \in \mathcal{U}\}$ for each finite $T \subseteq I$.

Assume a topological space $X$ and ultrafilter $\mathcal{U}$ on $I$. For each $x \in X$ and $f \in X'$, $x = \mathcal{D}$-limit $f$ if $\{i \mid f(i) \in V_x\} \in \mathcal{U}$ for each open set $V_x$ containing $x$.

Essentially the $\mathcal{D}$-lim operator on sequences $f \in X'$ acts like the limit operation on countable sequences, with the added property that, if $X$ is a compact space, $\mathcal{D}$-lim $f$ always exists. The properties of the $\mathcal{D}$-lim operator are listed below. A more complete treatment is provided in Chang and Keisler [3], pgs. 6-15.

Let $X$ be a topological space with basis $\delta$ (i.e., for each open set $V$, $V = \cup_{x \in \delta} 0$) and let $\mathcal{D}$ be an ultrafilter on $I$. $f$ will vary over elements of $X'$.

**D-L0** $\mathcal{D}$-lim $f$ is unique.

**D-L1** If $\{i \mid f(i) = f'(i)\} \in \mathcal{D}$, then $\mathcal{D}$-lim $f = \mathcal{D}$-lim $f'$ (or both don't exist).

**D-L2** $\mathcal{D}$-lim $(f_1, \ldots, f_n) = (\mathcal{D}$-lim $f_1, \ldots, \mathcal{D}$-lim $f_n)$.

**D-L3** $\mathcal{D}$-lim $f$ belongs to the closure (in $X$) of the range of $f$, (range $f = \{x \in X \mid \exists i \in I, f(i) = x\}$).

**D-L3'** The $\mathcal{D}$-limit of a constant sequence equals that constant value.

**D-L4** If $F$ is a continuous function from $X$ to $Y$, then $\mathcal{D}$-lim $F(f) = F(\mathcal{D}$-lim $f)$.

**D-L4'** If $R$ is a closed $n$-ary relation and $\forall i \in I, R(f_1(i), \ldots, f_n(i))$, then $R(\mathcal{D}$-lim $f_1, \ldots, \mathcal{D}$-lim $f_n)$.

**D-L5** If $X$ is a compact space, then $\forall f \in X'$ $\mathcal{D}$-lim $f$ exists.

**D-L6** If $\mathcal{D}$ is a regular ultrafilter on $I = S_n(\delta)$, $\delta$ a basis for $X$, then, for each set $S \subseteq X$ and limit point $x$, of $S$, there exists a sequence $f \in S'$ such that $\mathcal{D}$-lim $f = x$.

**D-L6'** If $X$ is dense in $\tilde{X}$, for each $x_0 \in \tilde{X}$, there exists an $X$-sequence $f \in X'$ such that $\mathcal{D}$-lim $f = x$, (i.e., the map $X' \xrightarrow{\mathcal{D}} \tilde{X}$ is onto).
TRANSFER THEOREMS FOR TOPOLOGICAL STRUCTURES

Fred Halpern

Transfer theorems are obtained for the following mathematical situations.

\( \mathcal{N} \) is a dense substructure of the compact structure \( \mathcal{X} \).
\( \{ \mathcal{N} \} \) is the set of all finitely generated substructures of \( \mathcal{X} \). \( F \) is a structure of functions from \( Y \) to the structure \( \mathcal{X} \).

The sentences transferred in the above situations are best described as “almost” positive, variables appearing in a negative subformula are quantified in a prescribed manner.

The main tools of this investigation are the manipulation of classical transfer theorems in the context of commutative diagrams, the ultraproduct construction, and the \( \mathcal{E} \)-limit operation of Chang and Keisler’s “Continuous Model Theory.”

Introduction. Every subgroup of an abelian group is abelian, any extension ring of a ring with zero divisors has zero divisors, and each homomorphic image of a commutative ring is commutative are instances of classical transfer theorems (Lemma 1). In this paper transfer theorems are obtained for the following mathematical situations:

\( \mathcal{X} \) is a dense substructure of the compact topological structure \( \mathcal{X} \).
\( \{ \mathcal{X} \} \) is the set of all finitely generated substructures of \( \mathcal{X} \), and \( F \) is a structure of functions from \( Y \) to the structure \( \mathcal{X} \).

The sentences transferred in the above situations are best described as “almost” positive, variables appearing in a negative subformula of the sentence are quantified in a prescribed manner.

The main tools of this investigation are the manipulation of the classical transfer theorems in the context of commutative diagrams, the ultraproduct construction, and the \( \mathcal{E} \)-limit operator popularized by Chang and Keisler [3].

Consider a first order language \( \mathcal{L} \) and its associated structures. A formula \( \sigma \) is identified with its prenex normal form, i.e., \( \sigma \) is assumed to be of the form

\[ Q_1 \varepsilon, Q_2 v_2, \ldots, Q_n v_n, \]

where each \( Q_i \) is a quantifier (\( \forall \) or \( \exists \)), each \( v_i \) is a variable, and \( M \) is a formula constructed from (positive) atomic formulae and their negations, negative atomic formulae, using the connectives “and”

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The formula $\sigma$ is $3$-generalized positive if the prefix of $\sigma$ begins with an initial string of existential quantifiers (i.e., $\sigma = \exists \bar{Q}, \bar{v}, \cdots, Q, v, M$) such that each negative variable of $\sigma$ is free or existentially quantified in the initial string $\exists \bar{v}$. The negation of a $3$-generalized positive formula is called a $\forall$-generalized negative formula: it begins with an initial string of universal quantifiers such that each of it nonnegative variables is free or universally quantified in the initial string.

**Theorem 1.** Let $\mathcal{N}$ be a dense substructure of the compact structure $\mathcal{M}$. Every $3$-generalized positive sentence satisfied by $\mathcal{N}$ is satisfied by $\mathcal{M}$.

This generalizes a result of Robinson [7].

**Theorem 2.** Let $\mathcal{N}$ compactify the collection of structures $\mathcal{M}$.

(i) Every $3$-generalized positive sentence satisfied in $\mathcal{M}$ is satisfied in $\mathcal{N}$.

(ii) Every $\forall$-generalized negative sentence valid in $\mathcal{M}$ is valid in $\mathcal{N}$.

**Corollaries to Theorem 2.** Let $\sigma$ be a $\forall$-generalized negative sentence (in an appropriate language).

2.1. $\sigma$ is valid for all extensions of the group $\mathcal{G}$ if $\sigma$ is valid for all compact groups which are extensions of $\mathcal{G}$.

2.2. $\sigma$ is valid for all total orderings if $\sigma$ is valid for all complete total orderings.

2.3. $\sigma$ is valid for all totally ordered lattices extending the ordering $\mathcal{F}$ if $\sigma$ is valid for all complete totally ordered lattices extending $\mathcal{F}$.

2.4. $\sigma$ is valid for all distributive lattices if $\sigma$ is valid for all complete distributive lattices.

2.5. $\sigma$ is valid for all boolean algebras if $\sigma$ is valid for all complete boolean algebras.

(ii) If, in addition, $\mathcal{N}$ is a compact structure, then every $3$-generalized positive sentence valid in $\{\mathcal{N}\}$ is satisfied by $\mathcal{M}$.

**Corollaries to Theorem 4.** Let $\sigma$ be a $\forall$-generalized positive sentence.

4.1. If $\sigma$ is valid for all finitely generated groups, then $\sigma$ is valid for all compact groups.

4.2. If $\sigma$ is valid for all finite total orderings (viewed as a lattice), then $\sigma$ is valid for all complete totally ordered lattices.
Books on Non-Standard Analysis are found at QA 299.82 ...

More Worthwhile References

1. Non-Standard Analysis in Practice by Francine Diener and Mark Diener


3. Infinitesimal Analysis of Curves and Surfaces by K.D. Stroyan in
   Handbook of Mathematical Logic edited by Jon Barwise (1976)