Integration by Guessing

The computations in two standard integration techniques, S*ubstitution* and *Integration by Parts,* can be streamlined by the *Integration by Guessing* approach. This method consists of three steps: Guess, Differentiate to check the guess, and then Adjust to get an exact fit. It is a simple and powerful methodology, useful in many situations.

The adjustments are of two kinds:
1. If the guess is off by a factor, divide by the factor.
2. If the derivative of the guess has an extra term, then subtract the integral of the term from the guess (One use is integration by parts).

Integration by Guessing emphasizes that *Substitution* and *Integration by Parts* begin with a hidden guess. When doing a substitution of variables, choosing a u comes along with a guess as to the form of the integral. Likewise, when integrating by parts, choosing functions u and dv corresponds to a guess. Recognizing the role of hidden guesses leads to shortcuts that simplify your work, avoiding tedious, distracting calculations.

To simplify exposition, all functions mentioned (including derivatives) are continuous. D denotes differentiation with respect to the variable x and we always assume Df exists. Df is also written as f ' or f '(x). $\int$ represents integration and after finding an anti-derivative, we write a final solution with arbitrary constant C. m and n denote positive integers. Thomas [1] is our basic reference for notation, concepts, and techniques.

1. **LUCKY GUESSES.** Many times you can make an inspired guess to find an indefinite integral. Your solution is justified by D F(x) = f(x) says the same thing as F(x) = $\int$ f(x)dx , i.e.,

Theorem A (Fundamental Theorem of the Calculus).
If D F(x) = f(x) , then $\int$ f(x)dx = *F(x) + C.*

Problem 1. Find $\int \frac{x}{\sqrt{1+x^2}}$ dx .

Tempted to use the substitution u $=1+x^2$ yielding $u^{-\frac{1}{2}}$ as integrand, expecting $u^{\frac{1}{2}} = \sqrt{u}$ as the solution, guess $\sqrt{1+x^2}$. Differentiating, D $\sqrt{1+x^2}$ = $\frac{x}{\sqrt{1+x^2}}$. So,

Solution 1. $\int \frac{x}{\sqrt{1+x^2}}$ dx = $\sqrt{1+x^2}$ + C.

As we all know, calculus is not easy Rarely does an inspired guess yield the correct solution exactly. Sections 2 and 3 illustrate how to adjust if the expression produced by a guess is close to what we want.

2. **GUESSES OFF BY A CONSTANT FACTOR .** If the expression produced by a guess is a multiple of what we want, we are rescued by the fact that D is a linear operator, i.e.,

Theorem B.

1. *If* $DF(x) = K f(x)$, *then* $\int \mathrm{f(x)dx} = \frac{F(x)}{K} + C$
2. *If* $DF(x) = \frac{f(x)}{K}$, *then* $\int \mathrm{f(x)dx} = K F(x) + C.$

Problem 2. Find $\int \mathrm{x}\sqrt{1+5x^2}\mathrm{dx}$.

Tempted to used substitution, $u = 1+5x^2$, yielding $u^{\frac{1}{2}}$ as integrand, expecting $u^{\frac{3}{2}}$, we guess $(1+5x^2)^{\frac{3}{2}}$. Since $D(1+5x^2)^{\frac{3}{2}} = 15x(1+5x^2)^{\frac{1}{2}}$. (K=15 in theorem B.1.

Solution 2. $\int \mathrm{x}\sqrt{1+5x^2}\mathrm{dx} = \frac{(1+5x^2)^{\frac{3}{2}}}{15} + \mathrm{C}.$

Problem 3. Find $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\mathrm{dx}$.

Guess $e^{\sqrt{x}}$. Since $De^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$. (K=2 in theorem B.2),

Solution 3. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\mathrm{dx} = 2e^{\sqrt{x}} + \mathrm{C}.$

The next problem illustrates guessing avoiding multiple fussy iterations of substitution.

Problem 4. Find $\int xe^{3x^2}\cos(2+5e^{3x^2})\mathrm{dx}$. Guess $\sin(2+5e^{3x^2})$.

Since $D\sin(2+5e^{3x^2}) = 30xe^{3x^2}\cos(2+5e^{3x^2})$.

Solution 4. $\int xe^{3x^2}\cos(2+5e^{3x^2})\mathrm{dx} = \frac{\sin(2+5e^{3x^2})}{30} + \mathrm{C}.$

3. **GUESSES OFF BY A SIMPLER FUNCTION.** If the produced expression has an extra term, use

Theorem C (D is a linear operator).

If $\mathrm{DF(x)} = \mathrm{f(x)} + R(x)$, *then* $\int \mathrm{f(x)dx} = F(x) - \int \mathrm{R(x)dx} + C.$

Recall that applying *Integration by Parts* requires finding appropriate u and dv where

Formula D (Integration by parts) **(**cf**.** Thomas [1], pp 547-549).

$$\int u\, dv = uv - \int v\, du\,.$$

3.A. INTEGRALS WITH INVERSE FUNCTIONS.

Rule 3.A. To find $\int x^n I(x)\mathrm{dx}$, guess $\frac{x^{n+1}}{n+1}I(x)$ (corresponding to u = I(x), dv = x^n dx).

Since $D\frac{x^{n+1}}{n+1}I(x)x^n = I(x) + \frac{x^{n+1}}{n+1}I'(x)$, $\int x^n I(x)\mathrm{dx} = \frac{x^{n+1}}{n+1}I(x)$ - $\frac{1}{n+1}\int x^{n+1}I'(x)dx$ + C.

Problem 5. Find $\int x \ln x\ \mathrm{dx}$.

Guess $\frac{x^2 \ln x}{2}$. Differentiating, $D\frac{x^2 \ln x}{2}$ = $x \ln x + \frac{x}{2}$.

Solution 5. $\int x \ln x\ \mathrm{dx} = \frac{x^2 \ln x}{2} - \int \frac{x}{2}dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4}$ + C.

Problem 6. Find $\int \arctan(x)\mathrm{dx}$

Guess $x\arctan(x)$. Differentiating, $Dx\arctan(x)$ = $\arctan(x) + \frac{x}{1+x^2}$.

Solution 6. $\int \arctan(x)\mathrm{dx} = x\arctan(x) - \int \frac{x}{1+x^2}dx = x\arctan(x) - \frac{\ln(1+x^2)}{2}$ + C.

3.B. OTHER INTEGRALS WITH MIXED FUNCTIONS.

Rule 3.B. To find $\int x^n E(x)\mathrm{dx}$, **guess** $x^n \bullet \int E(x)\mathrm{dx}$, corresponding to u = x^n, dv = E(x) dx

Rule 3.B works well when n=1 and E(x) is an *exponential type* function like

$$e^x,\ \sin x,\ \cos x,\ \sinh x,\ \text{and}\ \cosh x.$$

If n > 1, we get a reduction formula and the process must be iterated. *Tabular Integration* seems to work better **(**cf**.** Thomas [1], pp 552).

Problem 8. Find $\int x\cos(x)\mathrm{dx}$.

Note $\int \cos(x)\mathrm{dx} = \sin(x)$. Guess $x\sin(x)$. Since $Dx\sin(x)$ = $x\cos(x)$ + $\sin(x)$,

Solution 8. $\int x\cos(x)\mathrm{dx} = x\sin(x) - \int \sin(x)\mathrm{dx} = x\sin(x) - (-\cos(x))$ + C**.**

Problem 9. Find $\int xe^x dx$.

Guess xe^x. Differentiating, D xe^x = $xe^x + e^x$. So,

Solution 9. $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$.

4. INTEGRALS EQUAL TO AN *INVERSE TRIG FUNCTION*

Guessing also provides simple answers to integrals equal to a modified *Inverse Trig Function*, eliminating fussing with constants.

Theorem 4.1: Derived by guessing (Also note the nice pattern of answers)**.**

A. $\int \frac{1}{\sqrt{a^2 - b^2x^2}} dx = \frac{\arcsin(\frac{b}{a}x)}{b}$. Guess $\arcsin(\frac{b}{a}x)$

B. $\int \frac{1}{a^2 + b^2x^2} dx = \frac{\arctan(\frac{b}{a}x)}{ab}$. Guess $\arctan(\frac{b}{a}x)$

C. $\int \frac{1}{x \bullet \sqrt{b^2x^2 - a^2}} dx = \frac{\operatorname{arcsec}(\frac{b}{a}x)}{a}$. Guess $\operatorname{arcsec}(\frac{b}{a}x)$.

Example 4.1. Find $\int \frac{1}{a^2 + b^2x^2} dx$.

Guess $\arctan(\frac{b}{a}x)$. $\mathrm{D}\arctan(\frac{b}{a}x) = \frac{1}{1 + \frac{b^2x^2}{a^2}} \frac{b}{a} = \frac{a^2}{a^2 + b^2x^2} \frac{b}{a} = \frac{ab}{a^2 + b^2x^2}$,

Dividing our guess by ab yields the result.

The guessing technique can be used to prove theorems. This is illustrated in an appendix which contains a new proof of Pease [2] dealing with products of Trig (and Exponential) functions.

5. SUPLEMENTARY PROBLEMS.

We provide some extra problems. Hopefully, they are instructive and fun to do yourself.

Problem 4.1. Find $\int \frac{e^{5x}}{7 + 3e^{5x}} dx$.

Problem 4.2 Find $\int \frac{1}{9+4x^2}\mathrm{dx}$

***Hint 4.2*:** For $\int \frac{1}{a^2+b^2x^2}\mathrm{dx}$, guess $\arctan(\frac{b}{a}x)$.

Problem 4.3. Find $\int \ln x \; \mathrm{dx}$

Problem 4.4. Find $\int 2x\arctan(x)\mathrm{dx}$

Problem 4.5. Find $\int x\sin(x)\mathrm{dx}$.

Problem 4.6. Find $\int x^5 \ln x \; \mathrm{dx}$.

Problem 4.7. Find $\int (\mathrm{x}^x \ \ln\mathrm{x} + \mathrm{x}^x)\mathrm{dx}$.

***Hint 4.7*:** **Do not panic.** Explore the properties of x^x, e.g., $D\mathrm{x}^x = \mathrm{x}^x \ \ln\mathrm{x} + \mathrm{x}\cdot\mathrm{x}^{x-1}$.

REFERENCES

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1. R.L. Finney, F.R. Giordano, and M.D. Weir, Thomas' Calculus,10th ed., Addison Wesley Longman, Boston, 2001
2. 4. D.K. Pease, A useful integral formula, *Amer. Math. Monthly* **66** (1959) 908.

APPENDIX. PEASE'S THEOREM RESURRECTED.

Integrals of products of trig and exponential functions.

Definition 5.1: A function f(x) is *quasi-exponential* if $f'' = hf$ for some h. h is the *iteration strength* of f written $[f] = h$.

Theorem 5.2: *The following functions H(x) are quasi-exponential:*

A. $[e^x]=1,\ [\sin x]=[\cos x]=-1,\ [\sinh x]=[\cosh x]=1$ and $[x]=0$

B. $[\mathrm{H}(\mathrm{ax}+\mathrm{b})] = a^2[\mathrm{H}(\mathrm{x})]$.

C. In particular, $[e^{ax}]=a^2,\ [\sin ax]=[\cos ax]=-a^2,\ [\sinh ax]=[\cosh ax]=a^2$

Using Integration by parts, iterated twice, Pease [2] has shown

Theorem 5.3 (Pease): ***If*** *f and g are quasi-exponential with* $f''=hf$ *and* $g''=kg$ *(h≠k), then* $\int \mathrm{f}\cdot\mathrm{g} \ = \ \frac{f'\cdot g - \mathrm{f}\cdot g'}{h-k} + C = \frac{f'\cdot g - \mathrm{f}\cdot g'}{[f]-[g]} + C$.

Proof 5.3(By Guessing): There are two natural guesses: f'g and fg'.

Guess $f'\cdot g$. $Df'\cdot g = f''g + f'g'$. So, $Df'\cdot g = hfg + f'g'$ (since $f'' = hf$).
Guess $f\cdot g'$. $Df\cdot g' = fg'' + f'g'$. So, $Df\cdot g' = kfg + f'g'$ (since $g'' = kg$).

Which guess do we use ? We use both, taking a system of equations view.

$Df'\cdot g - Df\cdot g' = (h-k)\mathrm{f}\cdot g$ (subtracting guess computations). So

$f'\cdot g - f\cdot g' = (h-k)\int \mathrm{f}\cdot g$ (integrating).

Problem 5.4. Find $\int e^{ax}\sin(\mathrm{bx})\mathrm{dx}$.
Taking $f(x) = e^{ax}$ and $g(x) = \sin(bx)$: $f'(x) = ae^{ax}$, $g'(x) = b\cos(bx)$, $h = a^2$, $k = -b^2$
Solution 5.4. $\int e^{ax}\sin(\mathrm{bx})\mathrm{dx} = \dfrac{ae^{ax}\sin(\mathrm{bx}) - \mathrm{b}e^{ax}\cos(\mathrm{bx})}{a^2-(-b^2)}$.

Problem 5.5. Find $\int \sin(\mathrm{ax})\cdot\cos(b\mathrm{x})\mathrm{dx}$ $(a\neq b)$.
Taking $f(x) = \sin(ax)$ and $g(x) = \cos(bx)$: $f'(x) = a\cos(ax), g'(x) = -b\sin(bx), h = -a^2, k = -b^2$

Solution 5.5. $\int \sin(\mathrm{ax})\cdot\cos(\mathrm{bx})\mathrm{dx} = \dfrac{a\cdot\cos(\mathrm{ax})\cos(\mathrm{bx}) + b\cdot\sin(\mathrm{ax})\sin(\mathrm{bx})}{-a^2-(-b^2)}$

Note $\int \sin(\mathrm{ax})\cos(b\mathrm{x})\mathrm{dx}, \int \sin(\mathrm{ax})\cosh(b\mathrm{x})\mathrm{dx}, \int \sinh(\mathrm{ax})\cosh(b\mathrm{x})\mathrm{dx}$ are found similarly.

Corollary 5.6: *If f is quasi-exponential,* $\int \mathrm{x}\cdot\mathrm{f} = \dfrac{\mathrm{x}f' - f}{[f]} + C$

Problem 5.7. Find $\int \mathrm{x}\sin(\mathrm{ax})\,\mathrm{dx}$.

Solution 5.7. $[\sin(ax)] = -a^2$. So, $\int \mathrm{x}\sin(\mathrm{ax})\,\mathrm{dx} = \dfrac{ax\cdot\cos(\mathrm{ax}) - \sin(\mathrm{ax})}{-a^2}$ + C.