The Calculus of Quasi-Exponential Functions, Integration by Linear Combination.

We develop the technique of *integration by linear combination* which is useful in dealing with integrals involving *quasi-exponential functions* like e^x, $\sin(x)$, and $\sinh(x)$. We are motivated by Pease [5] which summarizes the results of iterated *integration by parts* computations to provide a simple formula for computing integrals like $\int e^x \sin(x)dx$ and $\int \sin(2x)\cos(3x)dx$.

Our main result generalizes Pease's formula to integrals of arbitrary products of *quasi-exponential functions* like $\int xe^x \sin^2(2x)\cos(3x)dx$. A new proof of his formula is derived by the *integration by guessing* technique of Halpern [3]. The guessing is best viewed as an instance of *integration by linear combination.*

Linear combination is taken over the Reals. A representative linear combination used is $Dfgh = f'gh + fg'h + fgh'$, easily derivable by logarithmic differentiation and even more clearly by Halpern [2].

To simplify exposition, all functions and their derivatives are continuous. D denotes differentiation with respect to the variable x and we always assume Df, also written $f'(x)$ or f', exists. Integration is denoted by $\int$ and after finding an anti-derivative, we write a final solution with arbitrary constant C. m and n denote positive integers and $a, b, c, \ldots$ denote real numbers. Thomas [1] is our basic reference for notation, concepts, and techniques.

1. QUASI-EXPONENTIAL FUNCTIONS.

Definition 1. A function $f(x)$ is *quasi-exponential* if $f'' = k_f f$ for some k_f, the *recurrence constant* of $f(x)$. If $f(x) \neq ax$, then it is *pure quasi-exponential.*

The basic quasi-exponential functions are $\sin ax$, $\cos ax$, $\sinh ax$, $\cosh ax$, *and* ax. Below are their recurrence constants. Basically, k_f takes on only two values, +1 and -1.

Lemma 1.1.
A. $k_{e^x} = k_{\sinh x} = k_{\cosh x} = 1$ and $k_{\sin x} = k_{\cos x} = -1.$ Also $k_x = 0.$
B. $k_{f(ax+b)} = a^2 k_{f(x)}$ for quasi-exponential $f(x).$
C. In particular, $k_{e^{ax}} = k_{\sinh ax} = k_{\cosh ax} = a^2$ and $k_{\sin ax} = k_{\cos ax} = -a^2.$
D. $k_f = k_{f'}.$

Using Integration by parts, iterated twice, Pease [5] has shown

Theorem 1.2 (Pease). *If f and g are quasi-exponential, then*

$$\int f \cdot g \quad = \quad \frac{f' \cdot g - f \cdot g'}{k_f - k_g} + C \quad (k_f \neq k_g).$$

Example 1.3. $$\int e^{ax}\sin(bx)dx \;=\; \frac{ae^{ax}\sin(bx) \;-\; be^{ax}\cos(bx)}{a^2 - (-b^2)}$$

Example 1.4. $$\int \sin(ax)\cdot\cos(bx)dx = \frac{a\cdot\cos(ax)\cos(bx) - (-b\cdot\sin(ax)\sin(bx))}{-a^2 - (-b^2)} \qquad a \neq b\ .$$

Corollary 1.5 (Pease Lite). *If f is quasi-exponential, then*

$$\int x \cdot f \quad = \quad \frac{xf' - f}{k_f} + C.$$

Example *1.6*. $$\int x\sin(ax)\, dx \;=\; \frac{ax\cos(ax) - \sin(ax)}{-a^2} + \mathrm{C}.$$

2. INTEGRATION BY LINEAR COMBINATION.

To introduce integration by linear combination:

2.1. *Proof of theorem 1.2 (by guessing).*

There are two natural guesses:

Guess $f' \cdot g$. $\quad Df' \cdot g = f''g + f'g'.$

Guess $f \cdot g'$. $\quad Df \cdot g' = fg'' + f'g'.$

So, replacing f'' and g'',

$$Df' \cdot g = k_f\, fg + f'g'$$
$$Df \cdot g' = k_g\, fg + f'g'$$
$$Df' \cdot g - Df \cdot g' = (k_f - k_g)\mathrm{f} \cdot g \qquad \blacksquare$$
$$f' \cdot g - f \cdot g' = (k_f - k_g)\int \mathrm{f} \cdot g.$$

Similarly,

Theorem 2.2. *If*

(2.2) $\qquad f'' = af,\ g'' = bg,$ and $h'' = ch,$ *then*

$$A. \qquad \int fgh = \frac{\begin{vmatrix} f'gh & 1 & 1 & 0 \\ fg'h & 1 & 0 & 1 \\ fgh' & 0 & 1 & 1 \\ f'g'h' & c & b & a \end{vmatrix}}{\begin{vmatrix} a & 1 & 1 & 0 \\ b & 1 & 0 & 1 \\ c & 0 & 1 & 1 \\ 0 & c & b & a \end{vmatrix}} + C, \qquad \begin{vmatrix} a & 1 & 1 & 0 \\ b & 1 & 0 & 1 \\ c & 0 & 1 & 1 \\ 0 & c & b & a \end{vmatrix} \neq 0.$$

$$B. \qquad \int fgh = \frac{(a-b-c)\, f'gh + (-a+b-c)\, fg'h + (-a-b+c)\, fgh' + 2f'g'h'}{a^2+b^2+c^2-2(ab+ac+bc)} + C$$

Proof 2.2.

Guessing $f'gh,\ fg'h,\ fgh',\ and\ f'g'h'$, by (2.2) ,

$$\begin{aligned} Df'gh &= afgh + f'g'h + f'gh' + 0 \\ Dfg'h &= bfgh + f'g'h + 0 + fg'h' \\ Dfgh' &= cfgh + 0 + f'gh' + fg'h' \\ Df'g'h' &= 0 + cf'g'h + bf'gh' + afg'h', \end{aligned}$$

written
$$\begin{pmatrix} a & 1 & 1 & 0 \\ b & 1 & 0 & 1 \\ c & 0 & 1 & 1 \\ 0 & c & b & a \end{pmatrix} \begin{pmatrix} fgh \\ f'g'h \\ f'gh' \\ fg'h' \end{pmatrix} = \begin{pmatrix} Df'gh \\ Dfg'h \\ Dfgh' \\ Df'g'h' \end{pmatrix}.$$

Cramer's rule produces fgh as a linear combination of $Df'gh,\ Dfg'h,\ Dfgh', and\ Df'g'h'$. Integrating yields theorem 2.2.

■

Theorem 2.2 remains valid when some strengths are equal. So, taking $h = g$ or $h = g'$ is allowed. $\int e^x \sin(x)\cos(x)dx,\ \int e^x \sin^2(x)dx,\ and \int \sin^3(x)dx$ are easily calculated. The calculation is facilitated by taking specialized variants of theorem 2.2 . Taking $c = b,\ h = g,$ etc. yield useful formulas. We omit the formulas for $\int fg^2, \int fgg'$, and $\int f^3$. Formulas for $\int f^2, \int f^2g^2, \int f^n$, and $\int f^2 f'^2$ are given in theorems 3.6 thru theorems 3.10 to illustrate additional technique. The below corollary indicates that the resolvent in theorem 2.2 is rarely 0.

Corollary 2.3. *If* $c = b$, *then*

$$\int fgh = \frac{(a-2b)\,f\,'gh\ -a\ fg'h\ -a\ fgh' + 2\,f\ 'g'h'}{a^2 - 4ab} + C.$$

Theorem 2.2 extends to integrals of the product of n quasi-exponential functions. Definition 2.4 is stated in terms of sequences to allow functions to be equal.

Definition 2.4. The sequence of functions $\langle f_1, f_2, \ldots, f_n \rangle$ *D-spans* $\langle F_1, F_2, \ldots, F_m \rangle$ if the derivative of each F_i is a linear combination of the functions in $\langle f_1, f_2, \ldots, f_n \rangle$. i.e., $F_i' = \sum_{j=1}^{n} a_{i,j} f_j$. If also $\langle F_1, F_2, \ldots, F_m \rangle$ *D-spans* $\langle f_1, f_2, \ldots, f_n \rangle$, we say $\langle f_1, f_2, \ldots, f_n \rangle$ *strongly D-spans* $\langle F_1, F_2, \ldots, F_m \rangle$.

Conjecture 2.5. *If* $\langle f_1, f_2, \ldots, f_n \rangle$ *strongly D-spans* $\langle F_1, F_2, \ldots, F_m \rangle$*, then the induced system of equations* $F_i' = \sum_{j=1}^{n} a_{i,j} x_j$ *has* $\langle f_1, f_2, \ldots, f_n \rangle$ *as its* ***unique*** *function solution.*

Definition 2.6. For a sequence functions $\langle f_1, f_2, \ldots, f_n \rangle$, each **formal** product $\tilde{f}_1 \cdot \tilde{f}_2 \cdot, \ldots, \tilde{f}_n$ is a *derived product* of $\langle f_1, f_2, \ldots, f_n \rangle$ where $\tilde{f}_i$ is either f_i *or* f'_i It is *odd* if the number of f'_i is odd; otherwise it is *even*.

Choosing some ordering of the derived products, $\langle \tilde{f}_1 \cdot \tilde{f}_2 \cdot, \ldots, \tilde{f}_n \mid odds \rangle$ denotes the list of odd derived products and $\langle \tilde{f}_1 \cdot \tilde{f}_2 \cdot, \ldots, \tilde{f}_n \mid evens \rangle$ denotes the list of even derived products . There are 2^n derived products, 2^{n-1} of them odd (even).

Theorem 2.7 (Main Theorem). *For* ***(pure)*** *quasi-exponential* f_i,

$$\langle \tilde{f}_1 \cdot \tilde{f}_2 \cdot, \ldots, \tilde{f}_n \mid evens \rangle \ \textbf{(strongly)}\ D-spans\ \langle \tilde{f}_1 \cdot \tilde{f}_2 \cdot, \ldots, \tilde{f}_n \mid odds \rangle.$$

As in theorem 2.2, theorem 2.7 yields a system of equations that (usually) computes $\int f_1 f_2 \ ,\ldots, f_n$.

Since listed functions can be equal, theorem 2.7 applies to products of powers of quasi-exponential functions . Therefore, we now consider $\int f_1^{e_1}, f_2^{e_2}, \ldots, f_n^{e_n}$, obtaining an analog of theorem 2.7. It reduces the size of the system of equations describing the integral, e.g., theorem 2.7 yields a system of 16 equations for $\int f^3 g^2$ whereas theorem 2.10 yields a system of only 6 equations.

Theorem 2.8. *For **pure** quasi-exponential* f,

A. $\left\langle \tilde{f}^{2n} \,|\, evens \right\rangle = \left\langle f^{2n-2i} \,|\, i = 0,...,n \right\rangle$ ***strongly*** $D-spans$ $\left\langle \tilde{f}^{2n} \,|\, odds \right\rangle = \left\langle f^{2n-2i-1} \,|\, i = 0,...,n-1 \right\rangle$.

B. $\left\langle \tilde{f}^{2n+1} \,|\, evens \right\rangle = \left\langle f^{2n+1-2i} f'^{2i} \,|\, i = 0,...,n \right\rangle$ ***strongly*** $D-spans$ $\left\langle \tilde{f}^{2n+1} \,|\, odds \right\rangle = \left\langle f^{2n-2i} f'^{2i+1} \,|\, i = 0,...,n \right\rangle$.

C. $\left\langle \tilde{x}^{2n} \,|\, evens \right\rangle = \left\langle x^{2n-2i} \,|\, i = 0,...,n \right\rangle$ $D-spans$ $\left\langle \tilde{x}^{2n} \,|\, odds \right\rangle = \left\langle x^{2n-2i-1} \,|\, i = 0,...,n-1 \right\rangle$.

D. $\left\langle \tilde{x}^{2n+1} \,|\, evens \right\rangle = \left\langle x^{2n+1-2i} \,|\, i = 0,...,n \right\rangle$ $D-spans$ $\left\langle \tilde{x}^{2n} \,|\, odds \right\rangle = \left\langle x^{2n-2i} \,|\, i = 0,...,n \right\rangle$.

Note the definition of *even (odd) derived products (of* f^n) in the theorem and that when n is even, there are more function variables than equations.

Definition 2.9. For a sequence function powers $\left\langle f_1^{e_1}, f_2^{e_2}, ..., f_n^{e_n} \right\rangle$, each **formal** product $\tilde{f}_1^{e_1}, \tilde{f}_2^{e_2}, ..., \tilde{f}_n^{e_n}$ is a *derived product* where $\tilde{f}_i^{e_n}$ denotes a choice of some derived product of $f_i^{e_i}$. It is *odd* if the number of odd choices is odd; otherwise it is *even*.

Theorem 2.10 (Generalized Main Theorem). *For **(pure)** quasi-exponential* f_i,

$$\left\langle \tilde{f}_1^{e_1}, \tilde{f}_2^{e_2}, ..., \tilde{f}_n^{e_n} \,|\, evens \right\rangle \text{ strongly } D-spans \left\langle \tilde{f}_1^{e_1}, \tilde{f}_2^{e_2}, ..., \tilde{f}_n^{e_n} \,|\, odds \right\rangle.$$

3. INTEGRATION BY LINEAR COMBINATION.

The above results use integration by linear combination implicitly. This section provides a more detailed description of the technique and provides concrete examples of its use. The material between lemma 3.3 and theorem 3.6 is interesting because it treats systems of equations where the number of variables and equations are different.

Loosely speaking, here is the first example of integration by linear combination:

Theorem 3.1 (To Integrate, Differentiate). *If* f *satisfies the (constant coefficient) linear differential equation* $y = \sum_{i=1}^{n} a_i D^i y$, *then* $\int f = \sum_{i=1}^{n} a_i D^{i-1} f + C$.

R. de la Llave [4] exploits this to integrate $\int e^{ax} \sin(bx) dx$ and (particularly neat) $\int x^n e^x \mathrm{dx}$. His introduction of operator algebra for these problems is interesting and merits further elaboration.

So, we now have three methods (Tabular integration (Thomas [1], pp552), by linear combination, and De la llave [4]) for integrating $\int e^{ax} \sin(bx) dx$. All three are based on the condition that iterated differentiation returns us to the original function, formalized by considering functions f that satisfy a linear differential equation $\sum_{i=0}^{n} a_i(x) D^i y = R(x)$. Theorem 3.13 supports this by showing our technique works with **every** constant coefficient case.

To simplify exposition, we restrict ourselves to products of two functions which satisfy differential equations of degree one or two. The extension of results to more than 2 functions is routine, see the proof of theorem 2.2. The restriction on degree is also routine and is described later.

The pool of generic linear equations is

Lemma 3.2.

A. $fg = \int f'g + \int fg'$.

B. $f'g = \int f''g + \int f'g'$.

C. $fg' = \int f'g' + \int fg''$.

D $f'g' = \int f''g' + \int f'g''$.

If f satisfies a differential equation of degree three, all similar products by f'' must be added to the list of basic linear equations. Higher degrees are handled similarly.

C_f (below) is not an arbitrary constant. It is determined by setting two integrals equal, $C_f = f'^2(a) - k_f f^2(a)$ for any a in the domain of f'. The non-dependence on a is justified by the proof of lemma 3.3A. Lemma 3.3C is not used further, but is of interest because it relates the two integrals. The constant C is arbitrary and depends on how the given integrations are performed.

Lemma 3.3.

A. $f'^2 = k_f f^2 + C_f$.

B. $D(ff') = k_f f^2 + f'^2$.

C. $ff' = k_f \int f^2 + \int f'^2 + C$.

Proof 3.3.

$Df^2 = 2ff'$ *and* $Df'^2 = 2f'f''$. Hence, since $f'' = k_f f$,

$Df'^2 = 2k_f ff'$. So,

$D(f'^2 - k_f f^2) = 0$. ■

Lemma 3.4.
$C_{\sin x} = C_{\cos x} = C_{\sinh x} = 1,\ C_{\cosh x} = -1\ , C_{e^x} = 0, C_{f'} = C_f,\ and\ C_{f(ax+b)} = a^2 C_{f(x)}.$

We next investigate the irregular system of equations given by

$$\langle ff' \rangle\ \textbf{\textit{strongly}}\ D-spans\langle f^2, f'^2 \rangle\ .$$

The results given in example 3.5 and theorem 3.6 seem to support conjecture 2.5.

Example 3.5. $\langle ff' \rangle\ D-spans\langle f^2, f'^2 \rangle$ yields $\begin{matrix} \mathrm{D}f^2 = 2ff' \\ \mathrm{D}f'^2 = 2k_f ff' \end{matrix}$.

It is solved for $\int ff'$ as explained in lemma 3.3.

$\langle f^2, f'^2 \rangle\ D-spans\langle ff' \rangle$ yields

Theorem 3.6. $\int f^2 = \dfrac{ff' - C_f x}{2k_f} + C.$

Proof 3.6.

$$\begin{aligned} D(ff') &= f'^2 + ff'' \\ &= f'^2 + k_f f^2 \\ &= C_f + k_f f^2 + k_f f^2 \quad \text{(by lemma3.A).} \end{aligned}$$

Integrating yields the theorem. ∎

Surprisingly, the above technique calculates a reduction formula for $\int f^n$. It is basically the usual derivation by integration by parts. Nevertheless, it is in interesting to have an explicit formula and a uniform calculation.

Theorem 3.7. $\int f^n = \dfrac{f^{n-1} f' - (n-1)C_f \int f^{n-2}}{nk_f} + C.\ \ (n \ge 2)$

Proof 3.7.

$$\begin{aligned} Df^{n-1}f' &= f^{n-1}f'' + (n-1)f^{n-2}f'^2 \\ &= k_f f^n + (n-1)f^{n-2}(k_f f^2 + C_f) \\ &= nk_f f^n + (n-1)C_f f^{n-2}, \end{aligned}$$

Integrating, $f^{n-1}f' = nk_f \int f^n + (n-1)C_f \int f^{n-2}.$ ∎

Emboldened by success in the above theorems

Theorem 3.8. $\int f^2g^2 = \dfrac{ff'g^2 - gg'f^2 + C_g\dfrac{ff'-C_fx}{2k_f} - C_f\dfrac{gg'-C_gx}{2k_g}}{2(k_f-k_g)} + C \quad (k_f \neq k_g).$

Proof 3.8. By theorem 2.10,
$\left\langle f^2g^2, f'^2g^2, f^2g'^2, f'^2g'^2, ff'gg' \right\rangle$ $D-spans$ $\left\langle ff'g^2, ff'g'^2, gg'f^2, gg'f'^2 \right\rangle.$
We only need to use the equations for $ff'g^2$ and $gg'f^2$, getting

$$D(ff'g^2 - gg'f^2) = f'^2g^2 + ff''g^2 - g'^2f^2 - gg''f^2$$

Replacing f'' and g'' by the quasi-exponential simplification and then using lemma 3.3 to replace f'^2 and g'^2 yields

$$D(ff'g^2 - gg'f^2) = (k_ff^2 + C_f)g^2 - (k_gg^2 + C_g)f^2 + (k_f - k_g)f^2g^2$$
$$= 2(k_f - k_g)f^2g^2 + C_fg^2 - C_gf^2.$$ Integration yields

$$ff'g^2 - gg'f^2 + C_g\int f^2 - C_f\int g^2 = 2(k_f - k_g)\int f^2g^2.$$

The theorem follows by replacing the left side integrals using lemma 3.6. ■

Theorem 3.8 is not valid when $k_f = k_g$, but we can get a partial result, finding $\int f^2f'^2$, based on the following pleasant surprise.

Lemma 3.9.
A. ff' *is* quasi-exponential.
B. $k_{ff'} = 4k_f$.
C. $C_{ff'} = C_f{}^2 = \left[f'^2 - k_ff^2\right]^2(a)$ for any a in the domain of f'.

Proof 3.9C. By lemma 3.3,

$(ff')'^2 = k_{ff'}(ff')^2 + C_{ff'}$. So,
$(f'^2 + k_ff^2)^2 = 4k_f(ff')^2 + C_{ff'}$. ■

One nice consequence is that $\int ff'gh$ is easily calculated using theorem 2.2. Also,

Theorem 3.10. $\int f^2f'^2 = \dfrac{ff'(f'^2 + k_ff^2) - C_f{}^2x}{8k_f} + C.$

Proof 3.10. By theorem 3.6 and then replacing $(ff')'$, $k_{ff'}$, and $C_{ff'}$ by lemmas 3.3 and 3.9,

$$\int f^2 f'^2 = \int (ff')^2 = \frac{ff'(ff')' - C_{ff'}x}{2k_{ff'}} + C. \qquad \blacksquare$$

$$= \frac{ff'(f'^2 + k_f f^2) - C_f{}^2 x}{2(4k_f)} + C.$$

To conclude, we explore computing $\int fg$ where f *and* g satisfy more complicated differential equations than $f'' = kf$. If an equation in f contains non-zero coefficients for f^i *and* f^j where *i-j* is odd, then we need

Lemma 3.11. *For **(pure)** quasi-exponential* f
$\left\langle \tilde{f}_1 \cdot \tilde{f}_2 \cdot ,\ldots, \tilde{f}_n \,|\, all \right\rangle$ (***strongly***) $D-spans$ $\left\langle \tilde{f}_1 \cdot \tilde{f}_2 \cdot ,\ldots, \tilde{f}_n \,|\, all \right\rangle$.

where $\left\langle \tilde{f}_1 \cdot \tilde{f}_2 \cdot ,\ldots, \tilde{f}_n \,|\, all \right\rangle = \left\langle \tilde{f}_1 \cdot \tilde{f}_2 \cdot ,\ldots, \tilde{f}_n \,|\, evens \right\rangle \bigcup \left\langle \tilde{f}_1 \cdot \tilde{f}_2 \cdot ,\ldots, \tilde{f}_n \,|\, odds \right\rangle$.

Our first example is

Theorem 3.12. *If* f *and* g *satisfy the (constant coefficient) linear differential equations*

$$f' = af + R(x)$$
$$g' = bg + S(x),$$

then $\int f \cdot g = \dfrac{fg - \int g \cdot R(x) - f \cdot S(x)}{a-b} + C.$

Proof 3.12. This is a simple case. We do not need expressions for $Dfg', Df'g,$ *or* $Df'g',$ only

$$Dfg = f'g + fg'.$$

Substituting f' *and* g' from the equations, we get

$$Dfg = (af + R(x))g + f(bg + S(x)) \qquad \blacksquare$$
$$= (a-b)fg + gR(x) + fS(x).$$

Against the objection to the integral $\int g\cdot R(x) - f\cdot S(x)$, we observe that sometimes the integration can be done, say $R(x) = g^m g' \text{ and } S(x) = f^n f'$.

Our last theorem below shows that our technique remains the same when extra coefficients appear in the differential equations.

Theorem 3.13. *If f and g satisfy the (constant coefficient) linear differential equations*

$$\begin{matrix} f'' = af' + bf + R(x) \\ g'' = cg' + dg + S(x), \end{matrix} \quad \textit{then}$$

$$\int fg = \frac{\begin{vmatrix} fg & 1 & 1 & 0 \\ fg' - \int fS(x) & c & 0 & 1 \\ f'g - \int gR(x) & 0 & a & 1 \\ f'g' - \int f'S(x) - g'R(x) & b & d & a+c \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 1 & 0 \\ d & c & 0 & 1 \\ b & 0 & a & 1 \\ 0 & b & d & a+c \end{vmatrix}} + C, \qquad \begin{vmatrix} 0 & 1 & 1 & 0 \\ d & c & 0 & 1 \\ b & 0 & a & 1 \\ 0 & b & d & a+c \end{vmatrix} \neq 0.$$

Proof 3.13.

By lemma 3.11, $\langle fg, fg', f'g, f'g' \rangle$ $D-spans$ $\langle fg, fg', f'g, f'g' \rangle$.

This motivates using **all** the equations of lemma 3.2. They yield, after substituting for f'' and g'',

A. $fg = \int f'g + \int fg'$.

B. $fg' = \int f'g' + \int f(cg' + dg + S(x))$.

C. $f'g = \int g(af' + bf + R(x)) + \int f'g'$.

D $f'g' = \int g'(af' + bf + R(x)) + \int f'(cg' + dg + S(x))$.

Algebraic manipulation yields the system of equations

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ d & c & 0 & 1 \\ b & 0 & a & 1 \\ 0 & b & d & a+c \end{pmatrix} \begin{pmatrix} \int fg \\ \int fg' \\ \int f'g \\ \int f'g' \end{pmatrix} = \begin{pmatrix} \text{fg} \\ \text{fg'} - \int fS(x) \\ f'g - \int gR(x) \\ f'g' - \int f'S(x) - g'R(x) \end{pmatrix}.$$

from which the theorem follows by Cramer's rule. ■

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