From Discrete to Analytic Inequality

Abstract: Mathematical folklore proposes that every discrete inequality has an analytic analog. We formalize this intuition with some theorems that have many classical inequalities as corollaries. Our sampling of *interesting* inequalities may be of independent value and some of our analytic analogs may be new.

Our motivation is that integrals are basically sums and therefore sum inequalities should be valid for integrals too. Applying the discrete inequality to Riemann sums does not work, it must be applied to "Riemann analogs", basically the sums without the interval length multiplier. A homogeneity condition is needed to convert the "analogs" to sums.

A **key insight** is we must start with a *very general inequality* valid for all n and all collections of n-sequences. The method also applies to inequalities where the sequence must satisfy special conditions (e.g., monotonic, convex) **if** the condition on functions imply that their Riemann analog sequences satisfy the analogous special condition.

0. INTRODUCTION

Mathematical folklore proposes that every discrete inequality has an analytic analog. We formalize this intuition with some theorems that have many classical inequalities as corollaries. Our sampling of *interesting* inequalities may be of independent value and some of our analytic analogs may be new.

Our motivation is that integrals are basically sums and therefore sum inequalities should be valid for integrals too. To my surprise, I found the situation is *slightly* more complicated. Applying the discrete inequality to Riemann sums does not work; it must be applied to "Riemann analogs", basically the sums without the interval length multiplier. A homogeneity condition is needed to convert the "analogs" to sums.

The method also applies to inequalities where the sequence must satisfy special conditions (e.g., monotonic, convex) **if** the condition on functions imply that their Riemann analog sequences satisfy the analogous special condition. A **key insight** is we must start with a *very general inequality* valid for all n and all collections of such n-sequences.

We assume a fixed interval $[a,\ b]$ and n is always a positive integer. $\vec{f}$ and $\vec{g}$ denote n-sequences. $\vec{f}=(f_1,...,f_n)$ *and* $\vec{g}=(g_1,...,g_n)$. f and g denote arbitrary (real valued) functions continuous on $[a,\ b]$ which implies that $\int_a^b f(t)\mathrm{dt}$ and $\int_a^b g(t)\mathrm{dt}$ always exist.

We consider inequalities of the form $R(X,Y) \leq S(X,Y)$ where X and Y are replaced by sums $\sum_{i=1}^{n} x_i$ and $\sum_{i=1}^{n} y_i$. R and S can not explicitly contain n since we take $\lim_{n\to\infty}$ to pass from sum to integral. Some discrete inequalities do contain n which must be replaced by $\sum_{i=1}^{n} 1$. Passing from discrete to analytic inequalities replaces n by $b-a$ because $\int_a^b 1\, dx = b-a$. To simplify statements and proofs we restrict ourselves to results for two variables, X and Y. It is clear that similar results are true for arbitrary n variables, $X_{1,} X_2 ... , X_n$. The Dominated convergence theorem and the density of continuous functions with compact support in $L^1[a,b]$ and $L^1(-\infty,+\infty)$ allows our results to be lifted to Lebesgue integrable functions (and hence to convergent infinite series of l^1).

Marcus and Minc [1] is our basic reference for inequalities and Stewart [2] is our basic reference for calculus notation and concepts.

Riemann sum for function $\int f(t)$ refers to the *right regular* *Riemann sum*

$$R_n = \sum_{i=1}^{n} f\left(a + \frac{i}{n}(b-a)\right) \cdot \frac{b-a}{n}.$$

Defining $a_i = a + \frac{i}{n}(b-a)$ and $\Delta_n = \frac{b-a}{n}$, $R_n = \sum_{i=1}^{n} f(a_i)\Delta_n$. Note that a_i depends on both i and n, but Δ_n depends only on n. Note also that $n\Delta_n = b-a$.

Definition 0.1. Given a function f, define ${}_n f_i$ $(i = 1, \ldots, n)$, the i-*Riemann analog of f*, by ${}_n f_i = f(a_i)$. So, $R_n = \sum_{i=1}^{n} {}_n F_i \, \Delta_n$.

Do not confuse ${}_n f_i$ with f_i. f_i is the i-th term of the sequence $\vec{f}$. Many of our proofs substitute ${}_n f_i$ for f_i in discrete inequalities.

Lemma 0.2. $\lim_{n\to\infty} \sum_{i=1}^{n} {}_n f_i \, \Delta_n = \int_a^b f(t)\mathrm{dt}.$

To illustrate our techniques and motivation, we state and prove theorem 0.3. Our main result is theorem 1.4.

Theorem 0.3. *Let* $S(X,Y)$ and $T(X,Y)$ *be continuous at* $\left(\int_a^b f(t)dt,\ \int_a^b g(t)dt\right)$. If for all $\vec{f}$ *and* $\vec{g}$

(0.4) $$S\left(\sum_{i=1}^{n} f_i, \sum_{i=1}^{n} g_i\right) \le T\left(\sum_{i=1}^{n} f_i, \sum_{i=1}^{n} g_i\right),$$

then $$S\left(\int_a^b f(t)dt,\ \int_a^b g(t)dt\right) \le T\left(\int_a^b f(t)dt,\ \int_a^b g(t)dt\right).$$

Proof 0.3. The proof exploits the extremely strong condition that the hypothesis inequalities hold for ***all*** pairs of n-sequences. Substituting $f_i = {}_n f_i\ \Delta_n$ *and* $g_i = {}_n g_i\ \Delta_n$ in (0.4),

$$\forall n\ \ \mathrm{S}\left(\sum_{i=1}^{n} {}_n f_i\ \Delta_n, \sum_{i=1}^{n} {}_n g_i\ \Delta_n\right) \le T\left(\sum_{i=1}^{n} {}_n f_i\ \Delta_n, \sum_{i=1}^{n} {}_n g_i\ \Delta_n\right).$$

Taking $\lim_{n\to\infty}$, the conclusion follows from lemma 0.2 by the continuity of S and T. ∎

Theorem 0.3 looks impressive, but it does not yield the classical inequalities which involve expressions like $\sum_{i=1}^{n} f_i g_i$ and $\sum_{i=1}^{n} f_i^2$. The substitution in (0.4) does not yield Riemann sums. We must add the condition that S and T are homogeneous of the same degree.

1. MAIN RESULTS. To simplify statements and proofs we state theorems for the case of two functions, $\varphi_1(x,y)$ and $\varphi_2(x,y)$, of 2 variables. It is clear that they are valid for m functions $\varphi_1,\ldots,\varphi_n$ of p variables $\varphi(x_1,\ldots,x_p)$.

Lemma 1.1 . $\varphi({}_n f_i, {}_n g_i)$ is the i-*Riemann analog of* $\varphi[f(x), g(x)]$.

Lemma 1.2 justifies ending a proof of an analytic inequality after establishing the corresponding Riemann sum inequality for arbitrary n. We assume that each integral $\int_a^b \varphi[f(t), g(t)]dt$ exists. So, by lemma 1.1,

Lemma 1.2. $\lim_{n\to\infty} \sum_{i=1}^{n} \varphi({}_n f_i, {}_n g_i)\ \Delta_n = \int_a^b \varphi[f(t), g(t)]dt,$

Definition 1.3. A function $S(X,Y)$ is *homogeneous of degree d* if $\forall t\ S(tX,tY) = t^d S(X,Y)$.

Theorem 1.4 (Main Theorem).

Let $S(X,Y)$ and $\mathrm{T}(X,Y)$ *be homogeneous functions of common degree d, continuous at* $(\varphi_1[\int_a^b f(t)\mathrm{dt},\int_a^b g(t)\mathrm{dt}],\varphi_2[\int_a^b f(t)\mathrm{dt},\int_a^b g(t)\mathrm{dt}])$. *If for all* $\vec{f}$, $\vec{g}$, *and n*

(1.5) $$S\left(\sum_{i=1}^{n}\varphi_1(f_i,g_i),\sum_{i=1}^{n}\varphi_2(f_i,g_i)\right)\le T\left(\sum_{i=1}^{n}\varphi_1(f_i,g_i),\sum_{i=1}^{n}\varphi_2(f_i,g_i)\right),$$ *then*

$$S\left(\int_a^b\varphi_1(f,g)\mathrm{dt},\int_a^b\varphi_2(f,g)\mathrm{dt}\right)\le T\left(\left(\int_a^b\varphi_1(f,g)\mathrm{dt},\int_a^b\varphi_2(f,g)\mathrm{dt}\right)\right).$$

Proof 1.4. Substituting the Riemann analogs ${}_nf_i$ and ${}_ng_i$ in (1.5) and multiplying by $\Delta_n^{\ d}$, by homogeneity,

$$\mathrm{S}\left(\sum_{i=1}^{n}\varphi_1({}_nf_i,{}_ng_i)\cdot\Delta_n,\sum_{i=1}^{n}\varphi_2({}_nf_i,{}_ng_i)\cdot\Delta_n\right)\le T\left(\sum_{i=1}^{n}\varphi_1({}_nf_i,{}_ng_i)\cdot\Delta_n,\sum_{i=1}^{n}\varphi_2({}_nf_i,{}_ng_i)\cdot\Delta_n\right).$$

and the theorem follows from lemma 1.2. ■

The discrete inequalities (1.7.A.1) and (1.7.B.1) are in Marcus and Minc [1], pp 108 and pp 109.

Corollary 1.7. *For all* $\vec{f}$ *and* $\vec{g}$

A. (**Holder inequality**). If p and q are positive with $\frac{1}{p}+\frac{1}{q}=1$.

1. $\sum_{i=1}^{n}f_ig_i\le\left(\sum_{i=1}^{n}|f_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|g_i|^q\right)^{\frac{1}{q}}$.

2. $\int_a^b f(t)g(t)dt\le\left(\int_a^b|f|^p(t)dt\right)^{\frac{1}{p}}\left(\int_a^b|g|^q(t)dt\right)^{\frac{1}{q}}$.

B. (**Minkowski inequality**). $\forall p\ge 1$.

1. $\left(\sum_{i=1}^{n}|f_i+g_i|^p\right)^{\frac{1}{p}}\le\left(\sum_{i=1}^{n}|f_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}|g_i|^p\right)^{\frac{1}{p}}$.

2. $\left(\int_a^b|f(t)+g(t)|^p\,dt\right)^{\frac{1}{p}}\le\left(\int_a^b|f|^p(t)dt\right)^{\frac{1}{p}}+\left(\int_a^b|g|^p(t)dt\right)^{\frac{1}{p}}$.

Proof of 1.7.A.2. Taking $\varphi_1(f_i,g_i)=f_ig_i$, $\varphi_2(f_i,g_i)=\ |f_i|^p$, and $\varphi_3(f_i,g_i)=|g_i|^q$, the inequality is of the form

$$\sum_{i=1}^{n} f_i g_i \le \left(\sum_{i=1}^{n} |f_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |g_i|^q\right)^{\frac{1}{q}}$$

$$X \le Y^{\frac{1}{p}} Z^{\frac{1}{q}}.$$

S and T are homogeneous of degree 1 since $t^{\frac{1}{p}} t^{\frac{1}{q}} = t^{\frac{1}{p}+\frac{1}{q}} = t^1$. ■

Proof of 1.7.B.2. Similarly, S and T are homogeneous of degree $\dfrac{1}{p}$. ■

All our proofs are based on theorem 1.4. Homogeneity allows us to multiply by $\Delta_n{}^d$, converting Riemann sums to integrals. It is easier to just do the algebraic manipulations suggested. Witness

*Second proof of 1.7.1***.**

$$\sum_{i=1}^{n} f_i g_i \le \left(\sum_{i=1}^{n} |f_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |g_i|^q\right)^{\frac{1}{q}}$$

$$\sum_{i=1}^{n} f_i g_i\ \Delta_n \le \left(\sum_{i=1}^{n} |f_i|^p\right)^{\frac{1}{p}} \Delta_n{}^{\frac{1}{p}} \left(\sum_{i=1}^{n} |g_i|^q\right)^{\frac{1}{q}} \Delta_n{}^{\frac{1}{q}}.$$ ■

Taking p=2 with q=2 in the Holder inequality and $p=2$ in the Minkowski inequality,.

Corollary 1.8.

A. **Cauchy–Schwarz inequality.** $\left[\int_a^b f(t)g(t)dt\right]^2 \le \int_a^b f^2(t)dt \cdot \int_a^b g^2(t)dt.$

B. **Triangle inequality** $(\int_a^b [f(t)+g(t)]^2 dt)^{\frac{1}{2}} \le (\int_a^b f^2(t)dt)^{\frac{1}{2}} + (\int_a^b g^2(t)dt)^{\frac{1}{2}}$

Xiang [3] provides more precise versions of the Cauchy–Schwarz inequalities. Its real content is lemma 1.9 with his clever, beautiful proof . Our restatement (theorem 1.12) of his analytic version is immediate from Theorem 1.4.

Lemma 1.9 (Xiang).

$$\left(\sum_{i=1}^{n} f_i g_i\right)^2 \le \left(\sum_{i=1}^{n}\max(f_i^2, g_i^2)\right)\left(\sum_{i=1}^{n}\min(f_i^2, g_i^2)\right) = \left(\sum_{i=1}^{n} f_i^2\right)\left(\sum_{i=1}^{n} g_i^2\right) - \left(\sum_{f_i^2 > g_i^2}(f_i^2 - g_i^2)\right)\left(\sum_{f_i^2 \le g_i^2}(g_i^2 - f_i^2)\right)$$

Definition 1.10. Define

A. $h_{f,g}^{+}(x) = \dfrac{f^2(x) - g^2(x) + \left|f^2(x) - g^2(x)\right|}{2}$ and $h_{f,g}^{+}(i) = \dfrac{f_i^2 - g_i^2 + \left|f_i^2 - g_i^2\right|}{2}$.

B. $h_{f,g}^{-}(x) = h_{g,f}^{+}(x)$ and $h_{f,g}^{-}(i) = h_{g,f}^{+}(i)$.

Lemma 1.11. $h_{f,g}^{+}(x)$ *is continuous.*

A. $h_{f,g}^{+}(x) = f^2(x) - g^2(x)$ *when* $f^2(x) \ge g^2(x)$ *and 0 otherwise.*

B. $\sum_{f_i^2 > g_i^2}(f_i^2 - g_i^2) = \sum_{i=1}^{n} h_{f,g}^{+}(i)$

C. $h^{+}({}_n f_i, {}_n g_i)$ is the i-*Riemann analog of* $h_{f,g}^{+}(x)$.

D. *Analogous results hold for* $h_{f,g}^{-}(x)$ **.**

Theorem 1.12 (Xiang).

$$\left(\int_a^b f(t)g(t)dt\right)^2 \begin{array}{l} \le \left(\int_a^b \max[f^2(t), g^2(t)]dt\right)\left(\int_a^b \min[f^2(t), g^2(t)]dt\right) = \\ \le \left(\int_a^b f^2(t)dt\right)\left(\int_a^b g^2(t)dt\right) - \left(\int_a^b h_{f,g}^{+}(t)dt\right)\left(\int_a^b h_{f,g}^{-}(t)dt\right). \end{array}$$

Proof 1.12. Restate lemma 1.9 as

$$\left(\sum_{i=1}^{n} f_i g_i\right)^2 \le \left(\sum_{i=1}^{n}\max(f_i^2, g_i^2)\right)\left(\sum_{i=1}^{n}\min(f_i^2, g_i^2)\right) = \left(\sum_{i=1}^{n} f_i^2\right)\left(\sum_{i=1}^{n} g_i^2\right) - \left(\sum_{i=1}^{n} h_{f,g}^{+}(i)\right)\left(\sum_{i=1}^{n} h_{f,g}^{-}(i)\right).$$

The above inequalities are homogeneous of degree 2. So, the theorem follows from lemma 1.11 via theorem 1.4. ■

2. DISCRETE INEQUALITIES VALID ONLY FOR SPECIAL SEQUENCES.

Definition 2.1. Given conditions $\mathcal{C}$ and c

1. $(\vec{\mathrm{f}}, \vec{\mathrm{g}})$ is a $\mathcal{C}$ *-sequence if* $(\vec{\mathrm{f}}, \vec{\mathrm{g}})$ satisfies $\mathcal{C}$, written $\mathcal{C}(\vec{\mathrm{f}}, \vec{\mathrm{g}})$

2. (f,g) is a c *-sequence if* (f,g) satisfies c, written $c(\mathrm{f},\mathrm{g})$

3. c *forces* $\mathcal{C}$ if $c(\mathrm{f},\mathrm{g})$ implies $\forall n\, \mathcal{C}({}_n\vec{F}_i, {}_n\vec{G}_i)$ for the Riemann analogs ${}_nF_i$ and ${}_nG_i$.

Many important classical inequalities apply to sequences satisfying special conditions c. We can derive analytic inequalities valid for functions that satisfy analogous conditions $\mathcal{C}$ by the methodology of theorem 1.4 when c *forces* $\mathcal{C}$.

Lemma 2.2. *If* c *forces* $\mathcal{C}$, then

(2.3) $\quad \mathcal{C}(\vec{\mathrm{f}},\vec{\mathrm{g}}) \Rightarrow \quad \mathrm{R}\left(\sum_{i=1}^{n}\varphi_1(f_i,g_i),\sum_{i=1}^{n}\varphi_2(f_i,g_i)\right) \le S\left(\sum_{i=1}^{n}\varphi_1(f_i,g_i),\sum_{i=1}^{n}\varphi_2(f_i,g_i)\right)$ implies

(2.4) $\quad \mathrm{R}\left(\sum_{i=1}^{n}\varphi_1({}_nf_i,{}_ng_i),\sum_{i=1}^{n}\varphi_2({}_nf_i,{}_ng_i)\right) \le S\left(\sum_{i=1}^{n}\varphi_1({}_nf_i,{}_ng_i),\sum_{i=1}^{n}\varphi_2({}_nf_i,{}_ng_i)\right).$

Lemma 2.3. (We will define the terms later). If f is positive (monotone, convex, concave), then the Riemann analog ${}_nF_i = \mathrm{f}(\mathrm{a}_i)$ is positive (monotone , convex, concave).

Recall that $n\Delta_n = b - a$ **.** It is used extensively in the rest of section 2.

The discrete versions of theorem 2.4 are from Marcus and Minc [1], pp 105-106.

Theorem 2.4 (Power Mean Strength and Convexity). *For* $0 < r < s$ *and* $1 \le p \le 2$.

A. $$\left(\frac{\sum_{i=1}^{n}|f_i|^r}{n}\right)^{\frac{1}{r}} < \left(\frac{\sum_{i=1}^{n}|f_i|^s}{n}\right)^{\frac{1}{s}}$$

B. $$\left(\frac{\int_a^b |f(t)|^r\,dt}{b-a}\right)^{\frac{1}{r}} < \left(\frac{\int_a^b |f(t)|^s\,(t)dt}{b-a}\right)^{\frac{1}{s}}$$

C. $$\frac{\sum_{i=1}^{n}|f_i+g_i|^p}{\sum_{i=1}^{n}|f_i+g_i|^{p-1}} \le \frac{\sum_{i=1}^{n}|f_i|^p}{\sum_{i=1}^{n}|f_i|^{p-1}} + \frac{\sum_{i=1}^{n}|g_i|^p}{\sum_{i=1}^{n}|g_i|^{p-1}}.$$

D. $\dfrac{\int_a^b |f(t)+g(t)|^p\, dt}{\int_a^b |f(t)+g(t)|^{p-1}\, dt} \le \dfrac{\int_a^b |f|^p(t)dt}{\int_a^b |f|^{p-1}(t)dt} + \dfrac{\int_a^b |g|^p(t)dt}{\int_a^b |g|^{p-1}(t)dt}$.

Proof 2.4.2. Substitute the Riemann analogs in (2.4.1) and rewrite as

$$\left(\frac{\sum_{i=1}^{n} {}_n f_i^r \Delta_n}{n\cdot\Delta_n}\right)^{\frac{1}{r}} \le \left(\frac{\sum_{i=1}^{n} {}_n f_i^s \Delta_n}{n\cdot\Delta_n}\right)^{\frac{1}{s}}.$$

■

Proof 2.4.4. Substitute the Riemann analogs in (2.4.3), R and S are homogeneous of degree 0. The theorem is immediate from theorem 1.4. ■

The proofs of theorems 2.5, 2.7, and 2.10 (below) are similar and therefore omitted. One can choose the style of the proof, 2.4.2 or 2.4.4.

Definition 2.5.

1. Function f is *monotone* on the interval $[a,b]$ if $\forall x, y \varepsilon [a,b]\ \ x < y \Rightarrow f(x) \le f(y)$
2. The n-sequence $(f_1,\dots,f_n)$ is *monotone* if $\forall i < n\ \ f_i \le f_{i+1}$.

Theorem 2.5 (Chebyshev's sum inequality). *For all monotone $\vec{f}$ and $\vec{g}$ and functions f and g*

1. $\dfrac{\sum_{i=1}^{n} f_i}{n} \cdot \dfrac{\sum_{i=1}^{n} g_i}{n} \le \dfrac{\sum_{i=1}^{n} f_i g_i}{n}$ (Marcus and Minc [1], pp ??)
2. $\dfrac{\int_a^b f(t)dt}{b-a} \cdot \dfrac{\int_a^b g(t)dt}{b-a} \le \dfrac{\int_a^b f(t)g(t)dt}{b-a}$.

Definition 2.6.

1. Function f is *convex* on the interval $[a,b]$ if $\forall x, y \varepsilon [a,b]$ and $\forall t \varepsilon [0,1]$

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$$

2. The n-sequence $(f_1,\dots,f_n)$ is *convex* if $\forall i < n\ \ 2f_i \le f_{i-1} + f_{i+1}$.
3. C*oncave* is defined by reversing $\le$ in (A) and (B) above.

Theorem 2.7 (Jensen inequality). *If φ Is a convex function, for all monotone $\vec{f}$ and $\vec{g}$ and functions f and g*

1. $\varphi\left(\frac{\sum_{i=1}^{n} f_i g_i}{\sum_{i=1}^{n} f_i}\right) \leq \frac{\sum_{i=1}^{n} f_i \varphi(g_i)}{\sum_{i=1}^{n} f_i}$ (Marcus and Minc [1], pp ??)

2. $\varphi\left(\frac{\int_a^b f(t)g(t)dt}{\int_a^b f(t)dt}\right) \leq \frac{\int_a^b f(t)\varphi(g(t))dt}{\int_a^b f(t)dt}$

3. *If φ Is a concave function, the above inequalities are reversed.*

Taking $\forall i \ f_i = 1$ and f the constant function 1 ($\forall t \ f(t) = 1$),

Corollary 2.8 (Jensen inequality). *If φ is a convex function,*

1. $\varphi\left(\frac{\sum_{i=1}^{n} g_i}{n}\right) \leq \frac{\sum_{i=1}^{n} \varphi(g_i)}{n}$ (Marcus and Minc [1], pp ??)

2. $\varphi\left(\frac{\int_a^b g(t)dt}{b-a}\right) \leq \frac{\int_a^b \varphi(g(t))dt}{b-a}$

We now provide an analytic version of the (logarithmic) arithmetic-geometric Mean inequality.

Theorem 2.9 (Arithmetic-Geometric Mean inequality). *For all positive $\vec{f}$,*

$$\sqrt[n]{f_1 \cdot f_2 \cdots f_n} \leq \frac{f_1 + f_2 + \cdots f_n}{n}.$$

Theorem 2.10 (Logarithmic AGM inequality). *For all positive $\vec{f}$* and *f*

1. $\frac{\sum_{i=1}^{n} \ln f_i}{n} \leq \ln\left(\frac{\sum_{i=1}^{n} f_i}{n}\right).$

2. $\dfrac{\int_a^b \ln(f(t))dt}{b-a} \le \ln\left(\dfrac{\int_a^b f(t)\,dt}{b-a}\right)$.

Corollary 2.11. (Analytic AGM inequality). Exponentiating both sides of 2.10.2,

$$e^{\frac{\int_a^b \ln(\mathrm{f}(t))\mathrm{dt}}{b-a}} \le \frac{\int_a^b \mathrm{f(t)dt}}{b-a}.$$

To better understand (2.11) we apply it to f(t)= e^t and f(t)= t.

Corollary 2.13.

1. For f(t)= e^t, $\quad e^{\frac{b+a}{2}} \le \dfrac{\mathrm{e}^b - \mathrm{e}^a}{b-a}$.

2. For f(t)=t,

$$\frac{b^b}{a^a} \le e^{b-a} e^{\frac{b^2-a^2}{2}}.$$

$$b\ln b - a\ln a \le (b-a) + \frac{b^2-a^2}{2}.$$

3. DOUBLE SUMS AND INTEGRALS.

Going from a double sum inequality to a double integral inequality is analogous. Our first result, theorem 3.1, is straightforward using the methods of the previous section. We consider the version of Lagrange's identity produced by replacing $\sum_{i=1}^{n}\left(\sum_{j=1}^{i}(f_i g_j - f_j g_i)^2\right)$ by $\dfrac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(f_i g_j - f_j g_i)^2\right)}{2}$.

Theorem 3.1 (Lagrange's Identity).

1. $\left(\sum_{i=1}^{n} f_i^{\,2}\right)\left(\sum_{i=1}^{n} g_i^{\,2}\right) - \left(\sum_{i=1}^{n} f_i g_i\right)^2 = \dfrac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(f_i g_j - f_j g_i)^2\right)}{2}$ (Marcus and Minc [1], pp 110)

2. $\left(\int_a^b f^2(t)dt\right)\left(\int_a^b g^2(t)dt\right) - \left(\int_a^b f(t)g(t)dt\right)^2 \le \dfrac{\int_{t=a}^{b}\left(\int_{s=a}^{b}[f(s)g(t) - f(t)g(s)]^2 ds\right)dt}{2}$

Proof 3.1. Substituting Riemann analogs **in** 3.1.1 and multiplying by $\Delta_n{}^2$, we get

$$\left(\sum_{i=1}^{n} {}_nF_i^2 \cdot \Delta_n\right)\left(\sum_{i=1}^{n} {}_nG_i^2 \cdot \Delta_n\right) - \left(\sum_{i=1}^{n} {}_nF_i \cdot {}_nG_i \cdot \Delta_n\right)^2 \le \frac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}({}_nF_i \cdot {}_nG_j - {}_nF_j \cdot {}_nG_i)^2 \cdot \Delta_n\right)\cdot \Delta_n}{2}.$$

Recall that the existence of $\iint_R$ is very strong and so we may calculate it using any sequence of partitions, in particular the totally regular ones.

Taking $\lim\limits_{n\to\infty}$ of both sides of the inequality and noting that the right hand side is a Riemann sum for $\iint_{\bar{R}} [f(s)g(t) - f(t)g(s)]^2 dsdt$ where $\bar{R} = [a,b]x[a,b]$ and replacing it by the iterated integral $\int_{t=a}^{b}\left(\int_{s=a}^{b}[f(s)g(t) - f(t)g(s)]^2 ds\right)dt$ yields 3.1.2. ■

Similarly, there is an analytic analog for the Binet-Cauchy identity (Marcus and Minc [1], pp ?)

$$\left(\sum_{i=1}^{n} e_i g_i\right)\left(\sum_{i=1}^{n} f_i h_i\right) - \left(\sum_{i=1}^{n} e_i h_i\right)\left(\sum_{i=1}^{n} f_i g_i\right) = \frac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(e_i f_j - e_j f_i)(g_i h_j - g_j h_i)\right)}{2}$$

Our next result theorem 3.7 (Minkowski's double sum inequality) needs more justification. The key is lemma 3.5 which generalizes Fubini's theorem.

Henceforth, we assume a region $\bar{R} = [a,b]x[c,d]$ and a function $f(s,t)$ continuous on $\bar{R}$. Note that $f(s,t)$ is uniformly continuous on R.

Definition 3.2.

A. $a_i = a + \frac{i}{m}(b-a)$, $c_j = c + \frac{j}{n}(d-c)$, $\Delta^a{}_m = \frac{b-a}{m}$, and $\Delta^c{}_n = \frac{d-c}{n}$.

B. The (i,j) – *Riemann analog* of $f(s,t)$ is

$${}_{(m,n)}F_{(i,j)} = f(a_i, c_j), \quad (i = 1, \ldots, \text{m}) \text{ and } (j = 1, \ldots, \text{n}).$$

C. The *Riemann sum* for $\iint_R f(s,t)dtds$ is $R_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} f(a_i, c_j)\Delta^c{}_n\, \Delta^a{}_m$.

When convenient, we will allow partitions to be indexed from negative integers as well yielding the *Riemann sum* for $\iint_R f(s,t)dtds$ is $R_{m,n} = \sum_{i=-m}^{m}\sum_{j=-n}^{n} f(a_i,c_j)\Delta^c{}_n\,\Delta^a{}_m$.

Lemma 3.3.

A.
$$\lim_{(m,n)\to(\infty,\infty)} \sum_{i=1}^{m}\sum_{j=1}^{n} f(a_i,c_j)\Delta^c{}_n\,\Delta^a{}_m = \iint_R f(s,t)dtds \text{ and}$$
$$\lim_{(m,n)\to(\infty,\infty)} \sum_{i=-m}^{m}\sum_{j=-n}^{n} f(a_i,c_j)\Delta^c{}_n\,\Delta^a{}_m = \iint_R f(s,t)dtds.$$

B. $$\lim_{m\to\infty}\sum_{i=1}^{m}\left(\lim_{n\to\infty}\sum_{i=1}^{n} {}_{(n,m)}F_{(i,j)}\Delta^c{}_n\right)\Delta^a{}_m = \int_{t=c}^{d}\left(\int_{s=a}^{b} f(s,t)dt\right)ds.$$

Theorem 3.4 (Fubini's theorem, (Stewart [2], pp 997)**).**

A. $$\iint_R f(s,t)dsdt = \int_{t=c}^{d}\left(\int_{s=a}^{b} f(s,t)ds\right)dt.$$

B. $$\lim_{(m,n)\to(\infty,\infty)}\sum_{i=1}^{n} {}_{(n,m)}F_{(i,j)}\Delta^a{}_m\Delta^c{}_n = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\lim_{m\to\infty}\sum_{i=1}^{m} {}_{(n,m)}F_{(i,j)}\Delta^a{}_m\right)\Delta^c{}_n.$$

Theorem 3.4B follows from Theorem 3.4A by lemma 3.3.

Lemma 3.5 below states that a doubly indexed limit can be replaced by iterated singly indexed limits. Fubini's theorem is immediate, taking $\varphi(t)=t$.

Lemma 3.5 (Iterated Integral justification). Given *f(s,t) and a function* $\varphi(t)$ *continuous on* $[p,q]$, *the image of* $[c,d]$ under the function $\int_{s=a}^{b} f(s,t)ds$, *then*

$$\lim_{(m,n)\to(\infty,\infty)}\sum_{j=1}^{n}\varphi\left[\sum_{i=1}^{m} f(a_i,c_j)\Delta^a{}_m\right]\Delta^c{}_n = \int_{t=c}^{d}\varphi[\int_{s=a}^{b} f(s,t)ds]dt.$$

Proof 3.5. The key step is lemma 3.6, a consequence of the uniform continuity of $f(s,t)$ on R.

Lemma 3.6 (Uniform Integrability). *Given f(s,t),* $\forall\varepsilon>0\ \exists M\ \forall m>M\ \forall t\in[c,d]$

$$\left|\int_a^b f\left[s,c_j\right]ds - \sum_{i=1}^{m} f\left(a_i,c_j\right)\frac{b-a}{m}\right| < \varepsilon.$$

Theorem 3.7 (Minkowski's double sum inequality).

$\forall p>1$ *with positive double sequence* $f_{i,j}$ *and function* $f(s,t)$ *continuous in the region* $[a,b]x[c,d]$

1. $\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m} f_{i,j}\right)^{p}\right)^{\frac{1}{p}} \le \sum_{i=1}^{m}\left(\sum_{j=1}^{n} f_{i,j}{}^{p}\right)^{\frac{1}{p}}$ (Marcus and Minc [1], pp 109)

2. $(\int_{t=c}^{d}[\int_{s=a}^{b} f(s,t)ds]^{p}dt)^{\frac{1}{p}} \le \int_{s=a}^{b}[\int_{t=c}^{d} f^{p}(s,t)dt]^{\frac{1}{p}}ds$

Proof 3.7 Substituting Riemann analogs **in** 3.7.1, multiplying both sides by $\Delta^{a}{}_{m}{}^{p\frac{1}{p}} = \Delta^{a}{}_{m}$ and $\Delta^{c}{}_{n}{}^{\frac{1}{p}}$, and taking $\lim_{(m,n)\to(\infty,\infty)}$ of both sides, we get

$$\lim_{(m,n)\to(\infty,\infty)}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m} {}_{(n,m)}F_{(i,j)}\,\Delta^{a}{}_{m}\right)^{p}\Delta^{c}{}_{n}\right)^{\frac{1}{p}} \le \lim_{(m,n)\to(\infty,\infty)}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} {}_{(n,m)}F_{(i,j)}{}^{p}\Delta^{c}{}_{n}\right)^{\frac{1}{p}}\Delta^{a}{}_{m}.$$

Applying lemma 3.5 to the left side with $\varphi(t)=t^{p}$ and applying lemma 3.5 to the right side with $\varphi(t)=t^{\frac{1}{p}}$ yields the theorem. ■

Young's Convolution Inequality is treated similarly, but we need some transfer theorems, namely theorems 4.1 and 4.3. In the rest of this section we assume integers $1 \le p,q,r \le \infty$ satisfying $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1.$

Theorem 3.8. (Young's Convolution Inequality). Suppose *f* is in $L^{p}(\mathbf{R}^{d})$ and *g* is in $L^{q}(\mathbf{R}^{d})$. Then **,**

$$\left(\int_{a}^{b}(f*g)^{r}(t)dt\right)^{\frac{1}{r}} \le \left(\int_{a}^{b} f^{p}(t)dt\right)^{\frac{1}{p}}\cdot\left(\int_{a}^{b} g^{q}(t)dt\right)^{\frac{1}{q}},\ \textit{restated as}$$

$$\left(\int_{s=a}^{b}\left(\int_{t=-\infty}^{\infty} f(t)g(x-t)dt\right)^{r} ds)\right)^{\frac{1}{r}} \le \left(\int_{a}^{b} f^{p}(t)dt\right)^{\frac{1}{p}}\cdot\left(\int_{a}^{b} g^{q}(t)dt\right)^{\frac{1}{q}}.$$

When convenient, we will allow partitions to be indexed from negative integers as well yielding

The *Riemann sum* for $\iint_{R} f(s,t)dtds$ is $R_{m,n} = \sum_{i=-m}^{m}\sum_{j=-n}^{n} f(a_i,c_j)\Delta^{c}{}_{n}\,\Delta^{a}{}_{m}$.

Lemma 3.9.

$$\lim_{(m,n)\to(\infty,\infty)} \sum_{i=-m}^{m} \sum_{j=-n}^{n} f(a_i, c_j)\Delta^c{}_n\, \Delta^a{}_m = \iint_R f(s,t)dtds.$$

R denotes the Reals. So, R^n denotes Euclidean n-space and R^+ denotes the non-negative Reals.

Definition 3.9. The *right regular step functions* $_{(n,m)}s_{(i,j)}$ *for function f on* $\bar{R} = [a,b]x[c,d]$ *is defined by*

$$_{(n,m)}s_{(i,j)}(x,y) = f\left(a_i, c_j\right),\ a_{i-1} \le x < a_i \text{ and } \mathrm{c}_{j-1} \le y < \mathrm{c}_j\ ,\ (i = 1,\ldots,\ \mathrm{m}) \text{ and } (j = 1,\ldots,\ \mathrm{n}).$$

Note that the step function values are just the Riemann analogs $_{(n,m)}F_{(i,j)} = f\left(a_i, c_j\right)$.

The *canonical* (finite support) *step functions for function f* are the *step functions* $_{(n,m)}s_{(i,j)} = f(i,j)$ for the intervals $R = [0,m]x[0,n]$. Note that $\Delta^a{}_m = 1$ and $\Delta^c{}_n = 1$ for these step functions.

Applying a universal inequality to its canonical step functions yields a universal discrete inequality. This paper examines when the universal discrete inequality can be used to derive the universal analytic inequality.

Applying Young's convolution Inequality to its canonical step functions yields lemma 3.10 below. We will show how it can be used to derive the full analytic inequality , theorem 3.8. Our proof is circular; we just want to illustrate the passage from discrete to analytic inequality. Perhaps a clear insightful proof of lemma 3.10 will be found.

Lemma 3.10. $\left(\sum_{j=1}^{m}\left(\sum_{i=1}^{n} f_i g_{j-i}\right)^r\right)^{\frac{1}{r}} \le \left(\sum_{i=1}^{m} f_i^{\,p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m} g_i^{\,q}\right)^{\frac{1}{q}}$

Proof 3.8. By theorems 4.1 and 4.3, it suffices to prove

Lemma 3.11. $\left(\sum_{j=1}^{m}\left(\sum_{i=1}^{n}(f(c_i)g(a_j - c_i)\Delta^a{}_m\right)^r \Delta^c{}_n\right)^{\frac{1}{r}} \le \left(\sum_{i=1}^{m} f_i^{\,p}\Delta^a{}_m\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m} g_i^{\,q}\Delta^c{}_n\right)^{\frac{1}{q}}$

4. INEQUALITY FOLKLORE IN LIGHT OF THE RIEMANN ANALOG METHOD.

Inequality folklore imparts practical advice:

1. When encountering a general discrete inequality, ask if there is an analytic analog.

2. When trying to prove an analytic inequality, ask if there is a discrete analog which is easier to prove.

A **key insight** is we must start with a *very general inequality* valid for all n and all collections of n-sequences satisfying a c-condition. Homogeneity is usually needed.

The discrete inequality usually will have a clear, insightful proof. Standard multi-variable extrema techniques , especially the method of Lagrange multipliers, usually provides a proof, [bbbb] for example. It should be noted that many important inequalities follow from each other, [bbbb] [bbbb]. Also noteworthy is the fact that the general analytic inequalities show that certain spaces have an inner product or a norm.

The Riemann analog method illustrated in this paper provides the analytic inequality for continuous functions on finite regions. This suffices to establish the inequality for arbitrary Lesbegue integrals by theorem 4.5, providing inequalities for $L^p(R^n)$ spaces and l^p *spaces.*

Lang [1] is our basic reference for notation and concepts for Lesbegue integrals and spaces. To simplify theorem statements, we assume a funct*ion* $\varphi(t)$ continuous on R (or R^+ when we deal only with positive functions f).

Lemma 4.1 (Dominated convergence theorem). *Let* $g \in L^1(R^n)$ *and* $< f_i >$ *be a sequence of* Lesbegue integrable functions *with* $\forall i \left| f_i \right| < g$. If $g \in L^1(R^n)$ *and* $< f_i >$ *be a sequence* that converges pointwise to $f \in L^1(R^n)$

then $$\int_R f_i \;\; L^1(R^n) - converges\ to \int_R f.$$

Lemma 4.2. *Let* $g \in L^1(R^n)$ *and* $< f_i >$ *be a sequence of* Lesbegue integrable functions *with* $\forall i \left| f_i \right| < g$. If $g \in L^1(R^n)$ *and* $< f_i >$ *be a sequence* that converges pointwise to $f \in L^1(R^n)$,, *then* $\int_{s=-\infty}^{\infty} \varphi[\int_{R^{n-1}} f_i(s,t)dt]ds \;\; L^1(R^n) - converges\ to \int_{s=-\infty}^{\infty} \varphi[\int_{R^{n-1}} f(s,t)dt]ds.$

$\int_R \varphi(f(t))dt$ and $\int_R \varphi[\int_{R^n} f(s,t)ds]dt$ are called *integral expressions.*

Definition 4.3. The function f is continuous with compact support if there exists a finite (compact) region $\bar{R} \subset R^n$ with f continuous on $\bar{R}$ and 0 elsewhere.

Lemmas 4.1 and 4.2 combined with the density of continuous functions with compact support in $L^1(R^n)$ yields

Theorem 4.3. *Let* $R(X,Y)$ and $S(X,Y)$ *be continuous at* $\left(\int_R f, \int_R g\right)$. If for all continuous

functions f *and* g *and finite regions* $\bar{R} \subset R^n$, $R\left(\int_R f, \int_R g\right) \le S\left(\int_R f, \int_R g\right)$,

then $$R\left(\int_R f, \int_R g\right) \le S\left(\int_R f, \int_R g\right).$$ for all Lesbegue integrable functions

f *and* g *and regions* $R \subset R^n$.

e combine and summarize the main results of this paper in

Theorem 4.4 (Inequality transfer to Lesbegue spaces theorem). *Let* $R(X,Y)$ and $S(X,Y)$ *be continuous at* $\left(\int_R f, \int_R g\right)$. If for all continuous functions

f *and* g *and finite regions* $R \subset R^n$, $R\left(\int_R f, \int_R g\right) \le S\left(\int_R f, \int_R g\right)$,

then $$R\left(\int_R f, \int_R g\right) \le S\left(\int_R f, \int_R g\right).$$ for all Lesbegue integrable functions

f *and* g *and regions* $R \subset R^n$.

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$\lim_{n\to\infty}\varphi\left[\lim_{m\to\infty}\sum_{i=1}^{m} f(a_i,c_j)\,\Delta^a{}_m\right]\Delta^c{}_n = \int_{t=c}^{d}\varphi[\int_{s=a}^{b} f(s,t)ds]dt,$ but more importantly

Definition 3.1. $(f*g)(x)=\int_a^b f(t)g(x-t)dt$ denotes the *convolution* of functions f and g.

Lemma 3.2. For *functions* f *and* g.

1. ${}_nF_i\,{}_nG_{n-i}$ is the i-*Riemann analog of* $f(t)g(b-t)$.
2. $\lim_{n\to\infty}\sum_{i=1}^{n} {}_nF_i\,{}_nG_{n-i}\;\Delta_n = \int_a^b (f*g)(t)dt.$

Corollary 2.9 (Jensen concave inequality). *If* φ *is a concave function,*

1. $\dfrac{\sum_{i=1}^{n}\varphi(g_i)}{n} \le \varphi\left(\dfrac{\sum_{i=1}^{n} g_i}{n}\right)$ (Marcus and Minc [1], pp ??)

2. $\dfrac{\int_a^b \varphi(g(t))dt}{b-a} \le \varphi\left(\dfrac{\int_a^b g(t)dt}{b-a}\right)$.

$$e^{\frac{\int_a^b \ln e^t \mathrm{dt}}{b-a}} \le \frac{\int_a^b \mathrm{e}^t \mathrm{dt}}{b-a}$$

$$e^{\frac{b^2-a^2}{2(b-a)}} \le \frac{\mathrm{e}^b - \mathrm{e}^a}{b-a}$$

Lemma 3.5 is all we need. Note that Lemma 3.5 its proof provides the bound $2(b-a)\varepsilon$ in more general situations: $\int_a^u f$ with u<b since u-a < b-a. arbitrary partitions P with $|P| < \delta$ of the proof.

Lemma 0.2. $\lim_{n\to\infty}\sum_{i=1}^{n} {}_nF_i \cdot \dfrac{b\text{-}a}{n} = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(a+\dfrac{i}{n}(b\text{-}a)\right)\cdot\dfrac{b\text{-}a}{n} = \int_a^b f(t)\mathrm{dt}.$

Riemann sum for f(s,t) refers to the <u>*Canonical*</u> *Riemann sum* $\sum_{i=1}^{n} f\left(a+\dfrac{i}{n}(b\text{-}a)\right)\cdot\dfrac{b\text{-}a}{n}$.

Also , since $\int_a^\infty \mathrm{f}(\mathrm{x})\mathrm{dx} = \lim_{b\to\infty}\int_a^b \mathrm{f}(\mathrm{x})\mathrm{dx}$, the above theorems are valid for b= ∞ .

I is the value of the d*efinite integral* of f over the interval [a,b]. The definition justifies considering the *definite integral* as the sum of infinitesimal values over [a,b].

Definition 2.1. The sequence a = $x_0 < x_1 ..., < x_n$ = b is a *partition* P of [a,b]. The *norm* $|P|$ of P is the largest value of $\Delta x_i = x_i - x_{i-1}$ (i=1,..., n). An δ- *partition* is a partition with norm less than δ ($\Delta x_i < \delta$ i=1,..., n).

Any analytic inequality has discrete analogs for all n by the use of **step functions**. In fact, the validity for associated step functions establish the validity for the analytic versions .

The analytic inequalities for intervals [a,b] extend to improper integrals and Lebesque integrals, as well as infinite sequences, in a standard manner. They say important things about vector product and Norms in Lp and lp spaces.

Proof of Thm 3.6

Wikipedia [4] is a good reference to fill in gaps in my proofs.

The sequence a = $x_0 < x_1 ..., < x_n$ = b is a *partition* P of [a,b]. The *norm* $|P|$ of P is the largest value of $\Delta x_i = x_i - x_{i-1}$ (i=1,..., n). An δ- *partition* is a partition with norm less than δ ($\Delta x_i < \delta$

Definition 3.3. Given a function f(t) , an interval $[a,b]$, and a partition P of $[a,b]$,

1. The sequence a = $x_0 < x_1 ..., < x_n$ = b is a *partition* P of [a,b]. The *norm* $|P|$ of P is the largest value of $\Delta x_i = x_i - x_{i-1}$ (i=1,..., n). Each $[x_{i-1} - x_i]$ is a *cell of* P.

2. $m_i = \dfrac{\min f(t)}{t \in [x_i, x_{i-1}]}$ and $M_i = \dfrac{\max f(t)}{t \in [x_i, x_{i-1}]}$

3. $L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i$, and $U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i$ denote the lower and upper Darboux sums.

Lemma 3.4. *For function f(t), partition P, and* sequence $c_i \in [x_i, x_{i-1}]$ (i=1,…, n),

If $\forall i\ |M_i - m_i| < \varepsilon$, then

1. $|U(f,P) - L(f,P)| < (b-a)\varepsilon$
2. $\left|\int_a^b f(t)dt - L(f,P)\right| \le (b-a)\varepsilon$ since $L(f,P) \le \int_a^b f(t)dt \le L(f,P)$, by 3.4.1.
3. $\left|\sum_{i=1}^{n} f(c_i)\Delta x_i - L(f,P)\right| \le (b-a)\varepsilon$ since $L(f,P) \le \sum_{i=1}^{n} f(c_i)\Delta x_i \le U(f,P)$, by 3.4.1.
4. $|\int_a^b f(t)dt - \sum_{i=1}^{n} f(c_i)\Delta x_i | < 2(b-a)\varepsilon$

Proof 3.4. 3.3.4 follows from the triangle inequality. ■

Lemma 3.6 (Uniform Integrability). Given *f(s,t),* $\forall \varepsilon > 0\ \exists M\ \forall m > M$

$$\forall j \le n,\ \left|\int_a^b f\left[s, c_j\right]ds - \sum_{i=1}^{m} f\left(a_i, c_j\right)\frac{b-a}{m}\right| < 2(b-a)\varepsilon.$$

Proof 3.5. $f(s,t)$ is uniformly continuous on R Thus $\forall \varepsilon\ \exists \delta$ for all partitions P with $|P| < \delta$, (x, c_j) and (y, c_j) belong to the same P-cell implies

$$\left|f(x, c_j) - f(y, c_j)\right| < \varepsilon.$$

Now choose an M with $\frac{1}{M} < \delta.\quad \forall m > M\ \forall i\ |M_i - m_i|\ < \varepsilon$ and 3.4.4 applies. ■

Lemma 3.6 (Iterated Integral justification). Given *f(s,t) and a function* $\varphi(t)$ *continuous on* $[p,q]$, *the image of* $[c,d]$ under the function $\int_{s=a}^{b} f(s,t)ds$, *then*

$$\lim_{(m,n)\to(\infty,\infty)} \sum_{j=1}^{n} \varphi\left[\sum_{i=1}^{m} f(a_i, c_j)\,\Delta^a{}_m\right]\Delta^c{}_n = \int_{t=c}^{d} \varphi[\int_{s=a}^{b} f(s,t)ds]dt.$$

Proof 3.6. We show $\forall(\varepsilon + (d-c)\varepsilon)\ \exists N\ \forall m,n > N$

(3.6.1) $\left|\sum_{j=1}^{n} \varphi\left[\sum_{i=1}^{m} f\left(a_i, c_j\right)\frac{b-a}{m}\right]\frac{d-c}{n} - \int_{t=c}^{d} \varphi[\int_{s=a}^{b} f(s,t)ds]dt\right| < \varepsilon + (d-c)\varepsilon.$

First, $\int_{s=a}^{b} f(s,t)ds$ exists for all t and is a continuous function of t. Hence $\int_{t=c}^{d} \varphi[\int_{s=a}^{b} f(s,t)ds]dt$ exists. So, $\forall\varepsilon\ \exists N_0 \forall n > N_0$ the normal n-partition P yields

(3.6.2) $$\left| \int_{t=c}^{d} \varphi[\int_{s=a}^{b} f(s,t)ds]dt - \sum_{j=1}^{n} \varphi[\int_{s=a}^{b} f\left(s,c_j\right)ds] \frac{d-c}{n} \right| < \varepsilon.$$

In view of (3.6.2), (3.6.1) follows from $\exists M > N_0\ \forall m > M$

$$\sum_{j=1}^{n} \left| \varphi\left(\int_a^b f\left(s,c_j\right)ds \right) - \varphi\left(\sum_{i=1}^{m} f\left(a_i,c_j\right) \frac{b-a}{m} \right) \right| \frac{(d-c)}{n} < (d-c)\varepsilon.$$

which would follow from $\exists M > N_0\ \forall m > M$

(3.6.3) $\forall j \le n, \left| \varphi\left(\int_a^b f(s,c_j)ds \right) - \varphi\left(\sum_{i=1}^{m} f\left(a_i,c_j\right) \frac{b-a}{m} \right) \right| < \varepsilon$

Proof 3.6.3. *$\varphi(t)$ is uniformly continuous on $\left[p,q\right]$,* so

$$\forall\varepsilon\ \exists\delta\ \forall t,t'\ \ |t-t'| < \delta \text{ implies } |\varphi(t) - \varphi(t')| < \varepsilon.$$

Now, for $\dfrac{\delta}{2(b-a)}$, lemma 3.5 provides an $M > N_0\ \forall m > M$

$$\forall j \le n,\ \left| \int_a^b f\left[s,c_j\right]ds - \sum_{i=1}^{m} f\left(a_i,c_j\right) \frac{b-a}{m} \right| < 2(b-a)\frac{\delta}{2(b-a)}.$$ So,

$$\forall j \le n,\ \left| \varphi\left(\int_a^b f(s,c_j)ds \right) - \varphi\left(\sum_{i=1}^{m} f\left(a_i,c_j\right) \frac{b-a}{m} \right) \right| < \varepsilon.\ \blacksquare$$

Young's inequality for convolutions[edit]

In real analysis, the following result is also called Young's inequality:[6]

Suppose f is in $L^p(\mathbf{R}^d)$ and g is in $L^q(\mathbf{R}^d)$ and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

with $1 \le p, q, r \le \infty$. Then

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Here the star denotes convolution, L^p is Lebesgue space, and

$$\|f\|_p = \left(\int |f(x)|^p \, dx \right)^{1/p}$$

denotes the usual L^p norm.

Theorem 3.8 (Binet-Cauchy Identity).

1. $\left(\sum_{i=1}^{n} e_i g_i\right)\left(\sum_{i=1}^{n} f_i h_i\right) - \left(\sum_{i=1}^{n} e_i h_i\right)\left(\sum_{i=1}^{n} f_i g_i\right) \leq \dfrac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(e_i f_j - e_j f_i)(g_i h_j - g_j h_i)\right)}{2}$ (Marcus and Minc [1], pp ???)

2. $\left(\int_a^b e(t)g(t)dt\right)\left(\int_a^b f(t)h(t)dt\right) - \left(\int_a^b e(t)h(t)dt\right)\left(\int_a^b f(t)g(t)dt\right) \leq \dfrac{\int_{t=a}^{b}\left(\int_{s=a}^{b}[e(s)f(t)-e(t)f(s)]\cdot[g(s)h(t)-g(t)h(s)]ds\right)dt}{2}$

1. $\left(\sum_{i=1}^{n} f_i^2\right)\left(\sum_{i=1}^{n} g_i^2\right) - \left(\sum_{i=1}^{n} f_i g_i\right)^2 \leq \dfrac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(f_i g_j - f_j g_i)^2\right)}{2}$ (Marcus and Minc [1], pp 110)

2. $\left(\int_a^b f^2(t)dt\right)\left(\int_a^b g^2(t)dt\right) - \left(\int_a^b f(t)g(t)dt\right)^2 \leq \dfrac{\int_{t=a}^{b}\left(\int_{s=a}^{b}[f(s)g(t)-f(t)g(s)]^2 ds\right)dt}{2}$

Proof 3.7. Substituting Riemann analogs **in** 3.7.1 and multiplying by $\left(\dfrac{b-a}{n}\right)^2$, we get

$$\left(\sum_{i=1}^{n} {}_nF_i^2 \cdot \frac{b-a}{n}\right)\left(\sum_{i=1}^{n} {}_nG_i^2 \cdot \frac{b-a}{n}\right) - \left(\sum_{i=1}^{n} {}_nF_i \cdot {}_nG_i \cdot \frac{b-a}{n}\right)^2 \leq \frac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}({}_nF_i \cdot {}_nG_j - {}_nF_j \cdot {}_nG_i)^2 \cdot \frac{b-a}{n}\right)\cdot \frac{b-a}{n}}{2}.$$

Taking $\lim_{n\to\infty}$ of both sides of the inequality and noting that the right hand side is a Riemann sum for $\iint_R [f(s)g(t)-f(t)g(s)]^2\,dsdt$ where $R=[a,b]\text{x}[a,b]$ and replacing it by the iterated integral $\int_{t=a}^{b}\left(\int_{s=a}^{b}[f(s)g(t)-f(t)g(s)]^2\,ds\right)dt$ yields 3.7.2. ■

We consider the version of Lagrange's identity produced by replacing

$\sum_{i=1}^{n}\left(\sum_{j=1}^{i}(f_i g_j - f_j g_i)^2\right)$ by $\dfrac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(f_i g_j - f_j g_i)^2\right)}{2}$.