Higher Order Derivatives of F(x,…,x)

Abstract: We provide an easy technique for computing the derivative and higher order derivatives of *expressions* f(x,…, x) *with many occurrences of x.* The main tool is the multivariate chain rule

The derivative of expressions f(x,…, x) *is the sum of the derivatives of the expression with respect to each occurrence of x.*

The derivatives of $\mathrm{f(x)g(x)h(x)}$, x^{x^x}, $\int_a^x f(x,t)dt$, and $\int_0^x \int_0^x f(u,v)dudv$ are easily found, as well as the higher order derivatives of the latter two functions. Leibnitz rule for $D^n(\mathrm{f(x)g(x)})$ is another easy consequence.

0. INTRODUCTION. We provide an easy technique for computing the derivative and higher order derivatives of *expressions* f(x,…, x) *with many occurrences of x.* The main tool is the multivariate chain rule

The derivative of expressions f(x,…, x) *is the sum of the derivatives of the expression with respect to each occurrence of x.*

The derivatives of $\mathrm{f(x)g(x)h(x)}$, x^{x^x}, $\int_a^x f(x,t)dt$, and $\int_0^x \int_0^x f(u,v)dudv$ are easily found, as well as the higher order derivatives of the latter two functions. Leibnitz rule for $D^n f(x)g(x)$ is another easy consequence.

We start with a function of n variables $f(u_{1,...,}u_m)$ and are interested in calculating $Df(x,...,x)$ and $D^n f(x,...,x)$. D represents differentiation with respect to the variable x and D_u represents partial differentiation with respect to the variable u. If there are less than two variables, we will use u and v. $f(u_1,\ldots,u_m)\ _{(u_1,\ldots,u_m)=(\mathrm{x},\ldots,\mathrm{x})}$ denotes substituting x for each of the variables u_i in f. So, $f(u_1,\ldots,u_m)\ _{(u_1,\ldots,u_m)=(\mathrm{x},\ldots,\mathrm{x})} = f(x,...,x)$..

Marsden and Tromba [2] is our basic reference for notation, concepts, and techniques. Justification in it for the basic results of elementary calculus are as follows: Lemma 0.2 **(**Theorem 1, pp 173), Theorem 1.1 **(**Theorem 11, pp 135-138)**.**

When we are interested in D^nf , we assume f is n-order differentiable in a region R,

Assumption 0.1. *All partial derivatives, including mixed partials, of degree less than or equal to n exist and are continuous in R.*

Lemma 0.2.

A. **(Equality of mixed partials).** $D_{u_i} \circ D_{u_j} = D_{u_j} \circ D_{u_i}$ *for all* $i, j \le m$.

B. (Binomial Expansion). $(D_u + D_v)^n = \sum_{i=0}^{n} \binom{n}{i} D_u{}^i D_v{}^{n-i}$

1. MAJOR THEOREMS.

Theorem 1.1 (Multivariate Chain rule).
Let $f(u_1,...,u_m)$ *be differentiable in region R and* $((u_1(x),...,u_m(x))$ *be a differentiable parametrization of a curve of R, then*

$$D\ \ f(x,...,x) = \sum_{k=1}^{m} \frac{\partial f}{\partial u_k} \times \frac{du_k}{dx}.$$

Writing $\frac{\partial f}{\partial u_k}$ as D_{u_k} and replacing $\frac{du_k}{dx}$ by 1,

Corollary 1.2 *If* $u_k(x) = x, k = 1,...,m$, *then*

A. $D\ \ f(x,...,x) = \left[\sum_{k=1}^{m} D_{u_k} f(u_1,...,u_m) \right]_{(u_1,...,u_m)=(x,...,x)}$

B. $D f(x,...,x) = \left[(D_{u_1} + \cdots + D_{u_m}) f(u_1,...,u_m) \right]_{(u_1,...,u_m)=(x,...,x)}$

Theorem 1.3. *If* $u_k(x) = x, k = 1,...,m$, *then*

A. $D^n f(x,...,x) = \left[(D_{u_1} + \cdots + D_{u_m})^n f(u_1,...,u_m) \right]_{(u_1,...,u_m)=(x,...,x)}$

B. (m=2) $D^n f(x,x) = \left[\sum_{i=0}^{n} \binom{n}{i} (D_u{}^i D_v{}^{n-i}) f(u,v) \right]_{(u,v)=(x,x)}$

Proof 1.3A. At first (1.3A) looks obvious by iteration of Corollary 1.2. Then one realizes that there is a substitution of variables at each iteration. The below simple induction on n examines the variable substitution and proves (1.3A). To clarify the main idea, we take m=2 and only prove $D^n f(x,x) = \left[(D_u + D_v)^n f(u,v) \right]_{(u,v)=(x,x)}$

For n=1, (1.3A) is just Corollary 1.2B.

For n>1, assume $D^n f(x,x) = \left[(D_u + D_v)^n f(u,v) \right]_{(u,v)=(x,x)}$.

Let F(x) = $\left[(D_u + D_v)^n f(u,v) \right]_{(u,v)=(x,x)}$,

$D^{n+1} f(x,x) = D(D^n f(x,x)) = D\left[(D_u + D_v)^n f(u,v) \right]_{(u,v)=(x,x)} = Df(x)$

$= \left[(D_u + D_v)((D_u + D_v)^n f(u,v) \right]_{(u,v)=(x,x)}$ by Corollary 1.2B. ∎

2. APPLICATIONS.

The following handy formula is easy to use. It is the main theorem of Halpern [1] where it provides easy of computation of the derivatives of f(x)g(x)h(x), x^{x^x}, $\int_a^x f(x,t)dt$, and $\int_0^x\int_0^x f(u,v)dudv$. Basically, when all $u_k(x)=x$ and we are taking partial derivatives, all u_k being held constant can be replaced by c to facilitate the differentiation, c being replaced by x afterwards.

Theorem 2.1.

$$D\ f(x,...,x) \ = \ Df(x,c,...,c)+Df(c,x,...,c)+,...,Df(c,c,...,x)\ _{c=x}$$

Application 2.2. $Dx^x = Dc^x + Dx^c = \ln c\cdot c^x + cx^{c-1} = \ln x\cdot x^x + x^x$

Application 2.3. *Start the below differentiations as indicated.*

A. $Dx^{x^x} = Dx^{c^c} + Dc^{x^c} + Dc^{c^x}$

B. $D\int_a^x f(x,t)dt = D\int_a^c f(x,t)dt + D\int_a^x f(c,t)dt$

C. $D\int_0^x\int_0^x f(u,v)dudv = D\int_0^c\int_0^x f(u,v)dudv + D\int_0^x\int_0^c f(u,v)dudv$

Application 2.4 (Leibnitz' rule for products).

$$D^n f(x)g(x) = \sum_{i=0}^{n}\binom{n}{i}D^i f(x)D^{n-i}g(x)$$

Proof 2.4.
View $f(x)g(x)$ as $f(u)g(v)$ with u=x and v=x. Since f(u) is a constant wrt v and each $D_v{}^j g(v)$ is a constant wrt u (j=1,…,n), theorem 1.3B yields

$$D^n f(x)g(x) = \left[\sum_{i=0}^{n}\binom{n}{i}(D_u{}^iD_v{}^{n-i})f(u)g(v)\right]_{(u,v)=(x,x)} = \left[\sum_{i=0}^{n}\binom{n}{i}\ D_u{}^i f(u)\quad D_v{}^{n-i}g(v)\ \right]_{(u,v)=(x,x)} \blacksquare$$

Applications 2.6. and 2.7 use the below lemma. Note the summand lower bound changing from i=0 to i=1 in (2.6) and (2.7) and the summand upper bound changing from i=n to i=n-1 in (2.7).

Lemma 2.5. For $i\geq 1$, $D_u{}^iD_v{}^j\int_a^u f(v,t)dt = D_u{}^{i-1}D_v{}^j f(v,u)$

Application 2.6. $D^n\int_a^x f(x,t)dt = [\int_a^x D_v{}^n f(v,t)dt]_{v=x} + \sum_{i=1}^{n}\binom{n}{i}[D_u{}^{i-1}D_v{}^{n-i}f(v,u)]_{(u,v)=(x,x)}$

Proof 2.6. Theorem 1.3B yields

$$D^n \int_a^x f(x,t)dt = \left[\sum_{i=0}^{n} \binom{n}{i} \left(D_u^i D_v^{n-i} \right) \int_a^u f(v,t)dt \right]_{(u,v)=(x,x)} = \left[\sum_{i=0}^{n} \binom{n}{i} D_u^i \int_a^u D_v^{n-i} f(v,t)dt \right]_{(u,v)=(x,x)}$$

$$= \left[\int_a^u D_v^n f(v,t)dt + \sum_{i=1}^{n} \binom{n}{i} D_u^{i-1} D_v^{n-i} f(v,u) \right]_{(u,v)=(x,x)} \quad \blacksquare$$

Application 2.7. $D^n \int_0^x \int_0^x f(s,t)dsdt =$

$$\left[\int_0^x D_v^{n-1} f(v,t)dt \right]_{v=x} + \left[\int_0^x D_u^{n-1} f(s,u)ds \right]_{u=x} + \sum_{i=1}^{n-1} \binom{n}{i} (D_u^{i-1} D_v^{n-i-1}) f(v,u) \Big]_{(u,v)=(x,x)}$$

Proof 2.7. Theorem 1.3B yields

$$D^n \int_0^x \int_0^x f(s,t)dsdt = \left[\sum_{i=1}^{n} \binom{n}{i} (D_u^i D_v^{n-i}) \int_0^u \int_0^v f(s,t)dsdt \right]_{(u,v)=(x,x)} =$$

$$\left[\int_0^u D_v^{n-1} f(v,t)dt + \int_0^v D_u^{n-1} f(s,u)ds + \sum_{i=1}^{n-1} \binom{n}{i} (D_u^{i-1} D_v^{n-i-1}) f(v,u) \right]_{(u,v)=(x,x)}$$

The first summand corresponds to i=0 and the second to i=n. ■

3. STATEMENTS FOR m>2.

Below, lemma 3.1 corresponds to lemma 0.2, theorem 3.2 corresponds to theorem 1.3, and application 3.3 corresponds to application 2.4.

Lemma 3.1. (Multinomial Expansion). $(D_{u_1} + \cdots + D_{u_m})^n = \sum_{k_1+\ldots+k_m=n} \binom{n}{k_1,\ldots,k_m} \prod_{1\le t\le m} D_{u_t}^{k_t}$

Theorem 3.2. *If* $u_k(x) = x, k = 1,\ldots,m$*, then*

$$D^n f(x,\ldots,x) = \left[\sum_{k_1+\ldots+k_m=n} \binom{n}{k_1,\ldots,k_m} \left(\prod_{1\le t\le m} D_{u_t}^{k_t} \right) f(u_1,\ldots,u_m) \right]_{(u_1,\ldots,u_m)=(x,\ldots,x)}$$

Application 3.3 (Extended Leibnitz' rule for products).
For m differentiable functions $f_t(x)$*, i=1,…,m,*

$$D^n \prod_{1\le t\le m} f_t(x) = \sum_{k_1+\ldots+k_m=n} \binom{n}{k_1,\ldots,k_m} \prod_{1\le t\le m} D^{k_t} f_t(x).$$

REFERENCES

---

1. F. Halpern, Using the multivariate chain rule, *Amer. Math. Monthly* **92** (1985) 144-145.

2. J. Marsden and A. Tromba, Vector Calculus,4th ed., W.H. Freeman, New York, 1996.