Distributive Normal form

Problem:

Show that for sets I and J and indexed collection of sets $\{S_{i,j} : i \in I, j \in J\}$, there exists a set J' and an indexed collection of sets $\{S_{i,j} : i \in I, j \in J'\}$ such that

$$\bigcap_{i\in I}\bigcup_{j\in J} S_{i,j} = \bigcup_{j\in J'}\bigcap_{i\in I} S_{i,j} = \bigcap_{i\in I}\bigcup_{j\in J'} S_{i,j}$$

Proof:

Set J' = J^I, the set of all functions from I to J and define $S_{i,f} = S_{i,f(i)}$ for each $f \in J^I$. We claim

$$\bigcap_{i\in I}\bigcup_{j\in J} S_{i,j} = \bigcup_{f\in J^I}\bigcap_{i\in I} S_{i,f} = \bigcap_{i\in I}\bigcup_{f\in J^I} S_{i,f}$$

The first equality is just the generalized distributive law. Now

if $x \in \bigcup_{f\in J^I}\bigcap_{i\in I} S_{i,f}$, *then* $\exists f_x$ *such that* $\forall i\ x \in S_{i,f_x}$. *So* $\forall i\ x \in \bigcup_{f\in J^I} S_{i,f}$. *So* $x \in \bigcap_{i\in I}\bigcup_{f\in J^I} S_{i,f}$

Conversely,

if $x \in \bigcap_{i\in I}\bigcup_{f\in J^I} S_{i,f}$, *then* $\forall i\ \exists f_{i,x}$ *such that* $x \in S_{i,f_{i,x}} = S_{i,f_{i,x}(i)}$.

Now define f *by* $f(i) = f_{i,x}(i)$.

$$x \in S_{i,f_{i,x}(i)}.\ So\ x \in S_{i,f(i)} = S_{i,f}.\ So\ \forall i\ x \in \bigcup_{f\in J^I} S_{i,f}.$$

Comments:

1. A similar theorem and construction works for $\bigcap_{i\in I}\bigcup_{j\in J_i} S_{i,j}$ for indexed sets J_i
2. If I and J are finite, a similar theorem holds in any Boolean algebra, in particular any first order logic (Note the concepts of Disjunctive (Conjunctive) Normal Form).
3. I first noticed this theorem while constructing a Craig interpolant in first order logic.