BI-MODEL COMPLETE THEORIES

by

Fred Halpern
10602 Stone Canyon Rd., #260
Dallas, Texas 75230

ABSTRACT

We define theory T as bi-model complete so as to extend the hierarchy: substructure complete, model complete. We show that T is bi-model complete iff every formula is equivalent to the lattice combination of existential and universal formulas relative to T. Also, some Cantor-Bernstein properties of a theory are investigated as strong versions of bi-model completeness.

Two preservation metatheorems used are of independent interest. The first generalizes Scott's interpolation theorem to characterize formulas preserved under simultaneous algebraic operators (Svenonius' definability theorem is immediate). The second concerns theories under which an algebraic operator becomes an (ostensibly) stronger operator.

Key words: Preservation theorem, model complete, Scott interpolation

Subject classification: 02H05

Special symbols: None

§ 0. INTRODUCTION: We define theory T as bi-model complete so as to extend the hierarchy: substructure complete, model complete. We show that T is bi-model complete iff every formula is equivalent to a lattice combination of existential and universal formulas relative to T. Cantor-Bernstein properties for a theory are investigated as strong forms of bi-model completeness.

Two preservation metatheorems used are of independent interest. The first generalizes Scott's interpolation theorem to characterize formulas preserved under simultaneous algebraic operators (Svenonius' definability theorem is immediate). The second concerns theories under which an algebraic operator becomes an (ostensibly) stronger operator.

Chang-Keisler [2] will serve as standard reference. Assume a first order language L without operations and a class of structures of appropriate type. The logical connectives and quantifiers: and, or, not, for all, and there exists are denoted (resp.) $\wedge$, $\vee$, $\sim$, $\forall$, $\exists$. Vector notation is used; $\sigma(v_1, \ldots, v_n)$ is written $\sigma(\overline{v})$ and $v_i \mathrel{R} w_i$ for $i = 1, \ldots, n$ is written $\overline{v} \mathrel{R} \overline{w}$.

Our results extend to languages with operations; occasionally, an example or application uses such languages.

§ 1. PRELIMINARIES:

Definition 1.1: An <u>algebraic situation</u> S is a diagram $A \overset{f}{\underset{g}{\rightrightarrows}} B$ where A and B are structures and f and g are relations (i.e., $f,g \subset A \times B$). Frequently, f or g is ignored giving a <u>simple algebraic situation</u> $A \xrightarrow{h} B$.

Consider the following types of <u>conditions</u>:

1. f (g) preserves the set of formulas F (G).
2. f (g) is a function (onto function).
3. f (g) is the inverse of a function (onto function).

Definition 1.2: A (algebraic) <u>description</u> D is a set of conditions. Situation S <u>satisfies</u> D is denoted $S \models D$ and S is a <u>model</u> of D. A description is <u>simple</u> if either there are no conditions on f or no conditions on g. A simple description can be viewed as a description of simple situations.

A description can be viewed as an <u>operator</u> which assigns the structure B to the structure A in each situation satisfying it.

Definition 1.3: Let D be a simple description for situations $A \xrightarrow{h} B$.

1. The sentence σ is <u>preserved under</u> D if $A \models \sigma$ implies $B \models \sigma$ for all situations satisfying D.

2. The formula $\sigma(\overline{v})$ is <u>preserved by</u> h <u>under</u> D if $A \models \sigma(\overline{a})$ implies $B \models \sigma(\overline{b})$ for all $\overline{a}$, $\overline{b}$ and situations satisfying D with $\overline{a}\ h\ \overline{b}$.

The generalized atomic formula method of Keisler [4] provides a sytactic description of the set of sentences and formulas preserved under D by each simple description.

To simplify notation, each set of formulas is assumed closed under $\sim$. In particular, L will denote the set of atomic formulas <u>and</u> their negations.

Definition 1.4:

1. The (H-substructure) H-extension (H-isomorphism) description is given by h is (the inverse of) a (onto) function preserving H.

2. The H-morphism description is given by h preserves H.

3. An existential (universal) H-formula is a formula constructed from H using $\wedge$, $\vee$, $\exists$ ($\wedge$, $\vee$, $\forall$).

4. A (H)-formula ([H]-formula) is a formula constructed from H using $\wedge$, $\vee$ ($\wedge$, $\vee$, $\forall$, $\exists$).

5. An existential (universal) H-formula is a $\exists$H- ($\forall$H-) formula.

For each designation * of formulas, we get the sets *-formulas and *-sentences. * will denote either set, meaning being clarified from context. Note the sets $\forall$H, $\exists$H, [H], and (H). When H=L, L is suppressed. Note the concepts: extension, substructure, isomorphism, morphism and the designations: $\forall$ and $\exists$. Also, (L) is the set of all quantifier free formulas and [L] is the set of all formulas. Define also $\forall\exists$ ($\exists\forall$) the set of all universal-existential (existential-universal) formulas.

Proposition 1.5: The below table matches simple descriptions to their sets of preserved formulas within logical equivalence.

	Description	Preserved Formulas
1.	H-extension	$\exists$H
2.	H-substructure	$\forall$H
3.	H-isomorphism	[H]
4.	H-morphism	(H)
5.	morphism	(L)

§ 2. PRESERVATION UNDER SIMULTANEOUS OPERATORS: A description D comes along with two simple descriptions: D(f) and D(g) induced by deleting (resp.) conditions on g (on f) from D. Sentences (formulas) preserved under D(f) are D(f)- sentences (formulas), and similarly for D(g).

The purpose of this section is to define some preservation concepts for a description D and characterize the resulting preserved formulas in terms of D(f)- and D(g)- sentences and formulas.

Definition 2.1: Let H and H' be sets of formulas. A (H, H')- formula is a formula constructed from $H \cup H'$ using $\wedge$ and $\vee$ (i.e., a lattice combination of H and H') and (H, H') is the set of all (H, H')- formulas.

Definition 2.2: Let $S: A \underset{g}{\overset{f}{\rightrightarrows}} B$ be a situation.

1. ${}_SP = \{\sigma : A \models \sigma \rightarrow B \models \sigma\}$.

2. ${}_SP_f = \{\sigma(\overline{v}): A \models \sigma(\overline{a}) \longrightarrow B \models \sigma(\overline{b}) \text{ if } \overline{a}\ f\ \overline{b}\}$.

3. ${}_SP_{f\cap g} = \{\sigma(\overline{v}): A \models \sigma(\overline{a}) \longrightarrow B \models \sigma(\overline{b}) \text{ if } \overline{a}\ f\ \overline{b} \text{ and } \overline{a}\ g\ \overline{b}\}$.

4. ${}_SP_{f,g} = \{\sigma(\overline{v}, \overline{w}): A \models \sigma(\overline{a}, \overline{a}') \longrightarrow B \models \sigma(\overline{b}, \overline{b}') \text{ if } \overline{a}\, f\, \overline{b} \text{ and } \overline{a}'\, g\, \overline{b}'\}$.

Definition 2.3: For a description D and theory T, and for * = f, g, $f\cap g$ and f,g;

1. $${}_{D(T)}P_* = \bigcap_{\substack{S \models D \\ A,B \models T}} {}_SP_* .$$

2. For $T = \emptyset$, abbreviate ${}_{D(T)}P_*$ as ${}_DP_*$.

(2.4): $\sigma \in {}_{D}P_{*} \left[{}_{D(T)}P_{*}\right]$ is <u>preserved</u> by * under D (relative to T),

THEOREM 2.5 (Simultaneous Operator Theorem): Within logical equivalence:

1. ${}_{D}P$ = (D(f) - sentences, D(g) - sentences).
2. ${}_{D}P_{f}$ = (D(f) - formulas, D(g) - sentences).
3. ${}_{D}P_{f\cap g}$ = (D(f) - formulas, D(g) - formulas).
4. ${}_{D}P_{f,g}$ = (D(f) - formulas in $\overline{v}$, D(g) - formulas in $\overline{w}$)

where $\overline{v} \cap \overline{w} = \emptyset$ and $\overline{v}$ and $\overline{w}$ are both infinite.

THEOREM 2.6 (Relativization):

$${}_{D(T)}P_{*} = \{\sigma : \exists\theta \in {}_{D}P_{*} \quad T \models \sigma \leftrightarrow \theta\}.$$

Theorems 2.5 and 2.6 are immediate by the methods of Halpern [3] (It is shown that in preserved formulas a variable does not occur in both F-formulas and G-formulas. The segregation of variables yields the lattice combinations by elementary logic, prefixes being dragged along). We proceed to corollaries of the theorems.

Since a description D is completely specified by D(f) and D(g), D will be denoted (D(f), D(g)).

COROLLARY 2.7 (Scott Interpolation):

1. A sentence σ is preserved under (F-isomorphism, G-isomorphism) iff σ is logically equivalent to a sentence of ([F] - sentence, [G] - sentence).

2. A formula σ is preserved by f under (F-isomorphism, G-isomorphism) iff σ is logically equivalent to a formula of ([F] - formula, [G] -sentence).

The above formula version of Scott interpolation will yield Svenonius' definability. The next corollary will yield the characterization of bi-model completeness.

COROLLARY 2.8: Let D = (F-substructure, G-extension).

1. $_{D}P$ = ($\forall$F - sentences, $\exists$G - sentences).
2. $_{D}P_{f}$ = ($\forall$F - formulas, $\exists$G - sentences).
3. $_{D}P_{f\cap g}$ = ($\forall$F - formulas, $\exists$G - formulas).

We provide a simple proof of Beth's definability theorem as an introduction to Svenonius' theorem. Adjoin two new n-ary relations $P(\overline{v})$ and $P'(\overline{v})$ to the language L yielding languages L(P, P') and L(P) and consider a theory T(P). If A is an L-structure and R an n-ary relation on A, (A, R) will denote the induced L(P)-structure. Formulas σ and θ are <u>equivalent relative</u> to theory T if $T \models \sigma \leftrightarrow \theta$.

PROPOSITION 2.9 (Beth Definability): The following are equivalent:

1. $T(P), T(P') \models \forall\overline{v}\ (P(\overline{v}) \leftrightarrow P'(\overline{v}))$, i.e., T(P) defines P implicitly.
2. If $(A, R), (A, R') \models T(P)$, then $R = R'$.
3. $P(\overline{v})$ is preserved under L-Isomorphism relative to T(P).
4. $P(\overline{v})$ is equivalent to an [L]-formula relative to T(P).
5. $T(P) \models \forall\overline{v}\ (P(\overline{v}) \leftrightarrow \theta(\overline{v}))$ for some [L]-formula $\theta(\overline{v})$, i.e.,

T(P) defines $P(\overline{v})$ explicitly.

PROOF: (3) $\leftrightarrow$ (4) follows from the characterization of formulas preserved under L-isomorphism (Proposition 1.5).

COROLLARY 2.10 (Svenonius Definability): The following are equivalent:

1. $(A, R), (A, R') \models T(P)$ and (A, R) is isomorphic to (A, R') imply $R = R'$.

2. $P(\overline{v})$ is preserved by f under (L-isomorphism, L(P)-isomorphism) relative to T(P).

3. $T(P) \models \forall \overline{v}\ (P(v) \leftrightarrow \bigvee_{1 \leq i \leq r} \alpha_i \wedge \beta_i(\overline{v}))$ for some integer r and r-tuples of [L]-formulas $\langle \beta_1, \ldots, \beta_r \rangle$ and [L(P)]-sentences $\langle \alpha_1, \ldots, \alpha_r \rangle$.

4. $T(P) \models \bigvee_{1 \leq i \leq n} \forall \overline{v}(P(v) \leftrightarrow \theta_i(\overline{v}))$ for some integer n and n-tuple of [L]-formulas $\langle \theta_1, \ldots, \theta_n \rangle$.

PROOF: $\bigvee_{1 \leq i \leq r} \alpha_i \wedge \beta_i(\overline{v})$ of (3) is the lattice combination of [L]-formulas and [L(P)]-sentences (written in normal form) gotten by applying the formula version of Scott interpolation (corollary 2.7 (2)) to (2).

(3) $\longrightarrow$ (4). Consider $T(P) \models \forall \overline{v}\ (P(\overline{v}) \leftrightarrow \bigvee_{1 \leq i \leq r} \alpha_i \wedge \beta_i(\overline{v}))$.

Define $s = \{1, \ldots, r\}$ is consistent if $T(P) \cup \{\alpha_i : i \in s\} \cup \{\sim \alpha_i : i \in s\}$ is consistent. Define $S = \{s : s \text{ is consistent}\}$.

$T(P) \models \bigvee_{s \in S} \forall \overline{v}(P(\overline{v}) \leftrightarrow \bigvee_{i \in s} \beta_i(\overline{v}))$. (4) $\longrightarrow$ (3). Define $\alpha_i = \forall \overline{w}\ P(\overline{w}) \leftrightarrow \theta_i(\overline{w})$. $T(P) \models \forall \overline{v}\ (P(\overline{v}) \leftrightarrow \bigvee_{1 \leq i \leq n} \alpha_i \wedge \theta_i(\overline{v}))$

A close look at the above proof shows that only the fact that the α_i are sentences is needed. The definability of $P(\overline{v})$ up to disjunction remains true even if we require the isomorphism of (1) to be with respect to additional relations (cf. Kochen [5]).

§ 3. THEORY STRENGTHENED DESCRIPTIONS AND BI-MODEL COMPLETENESS

Definition 3.1: Let D be a description and T a theory.

1. Situation S is a *model* of D(T) if S is a model of D with A, B $\models$ T.

2. *Elementary equivalence occurs under* D(T) if A and B satisfy the same sentences in each model of D(T).

3. f (f $\cap$ g) is an *elementary relation under* D(T) if f (f $\cap$ g) preserves all formulas in each model of D(T).

4. T is H-reductive (for sentences) if every formula (sentence) is equivalent to a formula (sentence) of H relative to T.

THEOREM 3.2:

1. Elementary equivalence occurs under D(T) iff T is ${}_D P$ reductive for sentences.

2. f (f $\cap$ g) is an elementary relation under D(T) iff T is ${}_D P_f$ (${}_D P_{f \cap g}$) reductive.

Note that h is an elementary relation under D(T) makes sense for simple descriptions D. Theorem 3.2 is true for simple descriptions, ${}_D P_f$ reductive being replaced by ${}_D P_h$ reductive.

COROLLARY 3.3:

1. If every homomorphism between T-structures is an isomorphism, then, every formula is equivalent to a positive formula relative to T.

2. Every formula of the theory of fields (linear order with primitives < and =) is equivalent to a positive formula.

COROLLARY 3.4:

1. If A, B $\models$ T and A and B are extensions of each other imply A and B satisfy the same sentences, then T is $(\forall, \exists)$ - reductive for sentences.

2. If every model of T is finite, then every formula is equivalent to the lattice combination of universal formulas and existential negative equality sentences relative to T (i.e., T is $(\forall, \exists\{\neq\}$ - sentences) reductive).

PROOF:

1. $(\forall, \exists)$ is the set of sentences preserved under (substructure, extension) by corollary 2.8 (1).

2. The result follows from corollary 2.8(2) because f between T-models must be an isomorphism in each (substructure, $\neq$ - extension).

DEFINITION 3.5: Theory T is bi-model complete if A, B $\models$ T, B is a substructure of A under $B \xrightarrow{f^{-1}} A$, and B is a extension of A under $A \xrightarrow{g} B$ imply $A \models \sigma(\overline{a}) \longrightarrow B \models \sigma(\overline{b})$ for all $\overline{a}$ f $\overline{b}$ and $\overline{a}$ g $\overline{b}$.

The following definition brings out the parallels between substructure, model, and bi-model completeness.

DEFINITION 3.6: Let T be a theory.

1. T is <u>substructure complete</u> if h is an elementary relation under morphism (T).

2. T is <u>model complete</u> if h is an elementary relation under extension (T).

3. T is <u>bi-model complete</u> if $f \cap g$ is an elementary relation under (substructure, extension) (T).

It is immediate from theorem 3.2 that

THEOREM 3.7:

1. T is substructure complete iff T is (L) - reductive (i.e., quantifier free formula reductive).

2. T is model complete iff T is $\exists$ - reductive.

3. T is bi-model complete iff T is $(\forall, \exists)$ - reductive.

PROOF: The reduction classes of formula are gotten as follows: (1) from Proposition 1.5 (5), (2) from Proposition 1.5 (1), (3) from corollary 2.8 (3).

Analgous to Robinson's prime formula test.

THEOREM 3.8: T is bi-model complete iff every $\forall\exists$ - formula is equivalent to a $(\forall, \exists)$-formula relative to T.

By Theorem 3.1.16 of Chang and Keisler [2] (every Σ_{n+1} sentence equivalent to a (Σ_n, π_n) - sentence).

THEOREM 3.9: Let $T \subset (\forall, \exists)$ be a theory. T is bi-model complete iff every $\exists\forall$- formula is equivalent to a $\forall\exists$- formula relative to T.

Every model complete theory is bi-model complete. Frequently, we can obtain a bi-model complete, non-model complete theory by removing uniquely defined constants from the language of a model complete theory. There are bi-model complete theories not obtainable in this manner.

EXAMPLE 3.10: The following theories are bi-model complete, but not model complete.

1. Theory of a single equivalence relation with an infinite number of classes, each class being infinite except for one class with exactly one member (cf. pg. 114 of Chang and Keisler [2]).

2. Theory of dense linear order with upper and lower endpoints in a language without constants.

3. Theory of a single equivalence relation with an infinite number of classes having exactly n members for each integer n.

Existential quantification out of the uniquely defined constants of a model complete theory need not yield a bi-model complete theory. Consider L = (<, =, a, b) and the theory in L that asserts a structure is an infinite linear order without endpoints, which excluding b, is a dense order, and in which a < b with no element between them.

COUNTEREXAMPLE 3.11: Let T be the theory obtained by adding the below axioms to the ($\forall, \exists$) theory of infinite linear order.

A1. $\forall x \quad \exists y, z \qquad y < x < z.$

A2. $\forall x \quad (x \leq a \quad \vee \quad b \leq x) \wedge \; a < b.$

A3. $\forall x, y, z \; \exists u \quad (x < y < z \longrightarrow x < u < y \qquad y < u < z).$

Let T' be the theory obtained by existential quantification of a and b in A2.

T is model complete, but T' is not bi-model complete (Let $M = (0, 1) \cup \{2, 3\} \cup (4, 5)$. $M \models T'$. In the language without constants, there exist embeddings $M \longrightarrow M$ which take (resp.) 2 to 3

(3 to 2). If T' were bi-model complete, 2 and 3 satisfy the same formula in M).

COUNTEREXAMPLE 3.11 lends support to the belief of Chang [1] that Lindstrom's theorem generalizes only for $T \subset \Pi_n$ since

COUNTEREXAMPLE 3.12: T' is ω-categorical without finite models and $T' \subset (\forall\exists, \exists\forall)$, but T' is not $(\forall, \exists)$ - reductive.

§ 4. CANTOR-BERNSTEIN PROPERTIES

DEFINITION 4.1: T has the (extended) Cantor-Bernstein property if for all models of (substructure, extension) with $A, B \models T$, there exists an isomorphism $h: A \longrightarrow B$ (with $f \cap g \subset h$).

THEOREM 4.2:

1. If T has the Cantor-Bernstein property, then elementary equivalence occurs under (substructure, extension) (T) and (thus) T is $(\forall, \exists)$ - reductive for sentences.

2. If T has the extended Cantor-Bernstein property, then T is bi-model complete and (thus) T is $(\forall, \exists)$ - reductive.

The concepts of (extended) Cantor-Bernstein property in cardinal number α is defined by considering only those situations with $|A|, |B| = \alpha$. A theorem analogous to the above for theories with no finite models can be proven by Skolem-Lowenheim considerations.

EXAMPLE 4.3: The following theories have the extended Cantor-Bernstein property.

1. Theory of pure equality.

2. Theory of vector spaces over a fixed field K.

3. Theory of algebraicly closed fields.

We thank Bruce Rose for suggesting a connection between ω_1 - categoricity and the Cantor-Bernstein property.

THEOREM 4.4: Let L be a countable language and T an ω_1-categorical theory in L with no finite models which is model complete.

1. T has the Cantor-Bernstein property.

2. T has the extended Cantor-Bernstein property in ω.

PROOF: Citations refer to Chang and Keisler [2]. By Morley's result that T is categorical in all uncountable powers, the case $|A|$ is uncountable is trivial. We need only prove (2).

Let $B \xrightarrow{f^{-1}} A$ and $A \xrightarrow{g} B$ be extensions with $|A|$, $|B| = \omega$ and A, $B \models T$. Let $f = (f^{-1})^{-1}$ and consider $A' = \{a \in A : \exists b \in B$ $a\ f\ b$ and $a\ g\ b\}$. A' is countable.

By corollary 7.1.9, we may assume that each $a \in A'$ is the interpretation of a constant of our language and $a\ f\ b$ (with $a\ g\ b$ too) means b is the interpretation in B of the same constant as a (Recall that g is really an elementary extension). Thus, we need only produce an isomorphism $h : A \longrightarrow B$ since the above will imply $f \cap g \subset h$.

Note that A is countable. Also, A is homogeneous by 7.1.27. Finally, since T is model complete, g is an elementary extension. Similar considerations hold for B and f^{-1}. Now consider the chain

(4.5) $A_1 \subset B_1 \subset A_2 \subset B_2 \ldots$

where A_i is isomorphic to A, B_i is isomorphic to B, $A_i \subset B_i$ is given by g properly interpreted, and $B_i \subset A_{i+1}$ is given by f^{-1} properly interpreted ($i = 1, 2, \ldots$).

PROPOSITION 3.2.9 applies to (4.5). Hence, $\bigcup_i A_i$ is isomorphic to A_1 and $\bigcup_i B_i$ is isomorphic to B_1. Clearly $\bigcup A_i = \bigcup B_i$, giving A is isomorphic to B.

CONJECTURE 4.6: If T is ω_1 - categorical with no finite models and T is bi-model (!) complete, then T has the extended Cantor-Bernstein property.

No analog of theorem 4.4 holds for ω - categorical theories.

COUNTEREXAMPLE 4.7: Let T be the theory of dense linear orderings without endpoints. Consider the sets of reals (0,2) and $(0,1) \cup (1,2)$. They are models of T which are extensions of each other, but they are not isomorphic.

Cantor-Bernstein properties should be considered in light of the following

THEOREM 4.8: If r is an elementary relation between A and B, then there exists an isomorphism $h : \overline{A} \longrightarrow \overline{B}$ between elementary extensions (resp.) of A and B such that $r \subset h$.

BIBLIOGRAPHY

1. C.C. Chang, Omitting Types of Prenex Formulas, J. Symb. Logic 32 (1967), 61-74.

2. C.C. Chang and H.J. Keisler, Model Theory, (North-Holland, Amsterdam 1973).

3. F. Halpern, Model Theory Via Proof Trees, (In Preparation).

4. H.J. Keisler, Theory of Models with Generalized Atomic Formulas, J. Symb. Logic 25 (1960), 1-26.

5. S.B. Kochen, Topics in the Theory of Definition, The Theory of Models, J.W. Addison, L. Henkin, and A. Tarski, eds. (North-Holland, Amsterdam), 265-273.

6. D. Scott, Interpolation Theorems, Proc. Third Intl. Cong. for Logic, Methodology and Philos of Science, Amsterdam, 1967. Talk was given but to my knowledge results were never published.